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Using projects in mathematics and engineering mathematics courses designed to stimulate learning

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Work Background / Experience: He interned at UNC/Chapel Hill, Argonne National Laboratory (Atomic Physics Division), and Entergy Corporation in Transmission and Distribution, and then Standards. He then began serving as a high school physics teacher for three (3) years where his students would inspire him to continue his education. Upon completing his doctoral studies, Dr. Moore began teaching Calculus- and Algebra-based Physics at Johnson C. Smith University in Charlotte, N.C. After two years as an Assistant Professor there, he began working at UAB in the School of Engineering, immediately addressing the leaky pipeline in the freshman and sophomore years by offering recitation courses in Calculus I, II, and III, while co-developing an Engineering Mathematics course with Dr. Gunter Stolz of the Mathematics Department. As the Co-Director for the Blazer BEST (Boosting Engineering, Science, and Technology) hub, he directly involves 800-1000 middle- and high-school students each year in the area of robotics.

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Using projects in mathematics and engineering mathematics courses designed to stimulate learning

Abstract

During the fall of 2008, an engineering mathematics course was developed at the University of Alabama at Birmingham to incorporate lessons in multivariable calculus and differential equations. The goal was to focus on topics with direct applicability in ensuing engineering courses, adding logical components, like units and dimensional analysis, tying mathematics and engineering together. The course added more of an engineering appeal to the traditional multivariable calculus and differential equations material with the use of engineering-based homework problems, test questions, and projects. The projects typically tackle problems in mechanics, electrical systems, population dynamics, optimizations, etc. designed to address the major focal areas of the course. This paper includes projects that tackle first-order ordinary differential equations (ODEs), second order ODEs, and multivariable calculus.

Introduction

With a year of planning between the School of Engineering and the Mathematics Department, a new four-hour course was developed to incorporate several science and engineering principles into the traditional mathematics topics of Calculus III and Differential Equations. The course is an Engineering Mathematics course that serves as an alternative track to the traditional Calculus III and Differential Equations courses. With four hours replacing seven, there were clearly topics that would be forfeited in order to make a cogent sequence of topics that serve the ensuing courses well. Three hours are returned to the departments so that students may take an elective course more aligned with their field, increasing their understanding of their chosen engineering field. Students are free to decide which track they will take. In pursuit of the topics for the course, all engineering faculty with Calculus III and/or Differential Equations as a pre- or co-requisite was interviewed to determine the needs of any and all upcoming courses.

It was with this understanding of the needs of our engineering students, as dictated by our faculty, that the topics of the course were developed. Also helpful were the discussions which would help to guide potential topics to pursue for projects. While the development of the projects is quite time consuming, they are priceless in developing a sense of intuition in the primary areas taught. The intent of this paper is to provide a set of projects that have been quite enlightening as the major areas of the course are taught – First-Order Ordinary Differential Equations, Second-Order Ordinary Differential Equations, and Multi-Variable Calculus.

Along with the previous three projects produced last year[1], the hope is that this dissemination will lead to greater use in other engineering mathematics courses and/or Calculus III and Differential Equations courses, increasing the pool of potential project choices – which does become an issue as students are very adept at finding solutions to overly used projects. While there are several wonderful projects that are in the journals, these should add to the current base of resources[2-6].

The use of projects allows students an opportunity to gain a greater sense of depth to vast breath of topics that are covered. Below are the topics covered in the course:

I. First-Order Ordinary Differential Equations (ODEs)
   A. Basic Concepts, Modeling
   B. Initial Value Problems
   C. Direction Fields
   D. Existence and Uniqueness
   E. Separable ODEs
   F. Linear ODEs
   G. Applications (primarily Biomedical, Mechanical, and Electrical)

II. Second-Order Ordinary Differential Equations
   A. Homogeneous Linear ODEs with constant coefficients
   B. Free Oscillations
   C. Forced Oscillations
   D. Electrical/Mechanical Systems

III. Multivariable Calculus
   A. Functions of Several Variables
   B. Partial Derivatives, Gradients, Directional Derivatives
   C. Line Integrals
   D. Multiple Integrals
   E. Spatial Transformations, Center of Mass, Moments

A cover page is included with each project outlining the expectations of the report. It is critical that students understand that working in groups is perfectly acceptable and encouraged. But, we sternly warn of any semblance of reports being alike. Students with projects that are too close for comfort are summarily failed for the assignment which represents 10-12% of their final grade, effectively dropping their final average one letter grade. Below is an example of the instructions given on the cover sheet for the first-order ordinary differential equations project.

> The project described below is self-contained, meaning that you should be able to do it by carefully reading through it and using what you learned in class about first order ordinary differential equations. A carefully written report is expected, which can be done in (legible) handwriting or typed with a text processor. You do not need to copy the problems into your report, but should clearly label to which problems your answers refer. Include the calculations which lead to your answers. Wherever appropriate, in particular if you are asked to state and justify an opinion, write your answers in full sentences and adequate English. Whenever numerical answers are required, find the exact values using a calculator. A certain amount of collaboration is acceptable in doing this project, but reports must be written up individually. Thus, when writing up your report, make sure that it is clearly different from reports of others. Reports which are virtually identical to others will not receive credit.

Project 1: Population Dynamics and Migration Effects

The Logistic Equation

The goal of this project is to use the logistic equation to predict the future population of two countries, Mexico and Germany. This will use available data on birth rates and death rates, and eventually also take into account migration effects. The two countries chosen can be considered as typical examples for quite opposite population trends. Mexico has much higher birth rate than death rate, but experiences population losses due to emigration. On the other hand, Germany has higher death rate than birth rate, but experiences population increases due to immigration. We will try to understand the long-term effects of these trends.

Recall from class that the logistic equation is given by

$$P'(t) = P(t)[a - bP(t)]. \tag{1}$$

If the initial value is $P(0) = P_0$, than the solution of the logistic equation is

$$P'(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}. \tag{2}$$

From this equation one can see that the population approaches the *carrying capacity*

$$P_\infty = \lim_{t \to \infty} P(t) = \frac{a}{b} \tag{3}$$

as the time *t* becomes large.

Throughout this project we will use the web site

indexmundi.com,

where population data for every country in the world for July 2008 may be found. Thus we will always consider July 2008 as time t = 0. The unit of time will be years.

Also, recall from class that *a* is the birth rate, which is the number of people born per year in a given country. The death rate is *bP*. At indexmundi.com we can find the death rates at $t = 0$. Thus we can determine the constant *b* in the logistic equation from

$$b = \frac{death\ rate}{P_0} \tag{4}$$

For example, on indexmundi.com the following information is found for the United States:

P_0 = 303,824,646   population in July 2008

birth rate = 14.18 births / 1000 population = 0.01418,

death rate = 8.27 deaths / 1000 population = 0.00827.

This gives $b = 2.72 \times 10^{-11}$ and the population would ultimately approach its capacity

$$P_\infty = \frac{a}{b} = 520{,}947{,}216.$$

We will understand later that this may not be a good prediction, because we did not include immigration/emigration effects in the model.

Problem 1:
(a) Use data from indexmundi.com to determine the population capacity (as predicted by the logistic equation) for Mexico and Germany. Also, in each case express P_∞ as a percentage of the current population P_0.
(b) In the homework folder on our course web site you will find the file project1tools.nbp which contains two Mathematica tools. Use the first of them to plot the population functions *P(t)* for Mexico and Germany over the next 200 years. You may also use MATLAB or wolframalpha.com. Add the plots as appendices to your project report. Comment on the differences.

(Note: The tool allows you to directly enter the death rate bP_0 instead of the tiny number for b.)

One of the observations from the plots in Problem 2(b) is that Mexico's population will grow very rapidly for a certain period of time. We next want to find the year when the most rapid growth occurs. This means we have to find the time t_M at which $P'(t)$ is maximal. According to the logistic equation (1) this happens when the function $f(P) = P(a - bP)$ is maximal.

Problem 2: Use Calculus to show that the function $f(P)$ is maximal for $P = \frac{a}{2b}$.

The result of Problem 2 says that the maximal population growth happens at the time t_M at which $P(t_M) = \frac{a}{2b}$ (exactly half of the maximal capacity).

Into this we can insert the formula (2) for $P(t_M)$. Now, with a bit of effort, we can solve the resulting equation to find t_M.

Problem 3: Find the year in which the population of Mexico grows the most before the logistic curve starts to flatten. How much does the population grow in that year?

The Effect of Migration
The population predictions from the previous section did not take the effects of immigration or emigration into account. We will now use a modified logistic equation as a *migration model*. It is given by the first order ODE

$$P'(t) = P(t)[a - bP(t)] + h. \qquad (5)$$

Here h is a constant. If $h > 0$, then it represents the *migration gain*, i.e. the number of people who immigrate into a country per year. If $h < 0$, then its absolute value is the *migration loss*, meaning the number of people who emigrate out of the country per year. The migration model (5) is still unrealistic because it doesn't take into account that migration rates change over time. But (5) should at least give better predictions than the basic logistic model (1).

The value of h for any given country can also be found on indexmundi.com by searching for the *net migration rate*. For example, for the United States one finds that the net migration rate currently is

$$\frac{2.92 \text{ migrants}}{1000 \text{ population}} = 0.00292.$$

To get h from this one has to multiply the net migration rate with the total population at time $t=0$, i.e. for the US we find

$$h = P_0 \times 0.00292 = 303{,}824{,}646 \times 0.00292 = 321{,}070$$

immigrants per year.

In principle, one can write down a formula for the solution of (5), but it will be even more complicated than (2). We instead use a different approach to see the effects of migration on the long time population development of a country. For the migration model (5) the solution $P(t)$ will still approach a carrying capacity $P_{\infty,m}$ as $t \to \infty$. We use the additional subscript m (for "migration") to distinguish these capacities from the values P_∞ found without considering migration. One can find the value of $P_{\infty,m}$ as follows:

If $P(t)$ would ever reach P_∞ it would have to stay constant from then on as birth, death and migration effects cancel each other out. This means that $P'(t) = 0$ and therefore we get from the right hand side of (5) that

$$P(a - bP) + h = 0.$$

Knowing a, b and h one can solve this quadratic equation and find two values for P. For all our examples, the larger one of these two values will be $P_{\infty,m}$ (this generally holds for the migration model as long as h does not become extremely large).

Problem 4: Using the migration model and data from indexmundi.com, find the carrying capacities $P_{\infty,m}$ of the United States, Mexico and Germany. Be sure to correctly interpret the net migration rates from indexmundi.com in determining the sign of h for each country. For all three countries, express the carrying capacity $P_{\infty,m}$ as a percentage to the capacities P_∞ found with the basic logistic model. Comment on the changes and the reasons for these changes. Also compare $P_{\infty,m}$ with P_0.

The second Mathematica tool in the file project1tools.nbp is a numerical solver for the migration model (5). After entering the data a, bP_0, h and P_0 it numerically solves (5) and plots the solution. The time interval and P-interval of the plot can also be chosen.

Problem 5: Use the second tool to find the population graph of Mexico based on the migration model (5) and with current data from indexmundi.com. Print out this graph and attach it to your report. Based on this graph, what is the approximate population of Mexico in 2100? What is the percentage change compared to Mexico's population in 2100 without taking migration into account (i.e. the result of Problem 1)?

Note: Choose the t and P-ranges appropriately. In particular, you may want to "zoom" the P-axis to get a good reading of the size of the population in 2100.

Changing birth and death rates

Here is an article that appeared in October 2008 on Google News.

Spain needs 100,000 qualified foreign workers, study finds.

AFP (10/23) reports, "Despite a slowing economy, Spain needs 100,000 qualified foreign workers per year until 2012 due to a shortage of IT, health and other professionals," according to a study from Etnia Communication. "In total the country will need between 250,000 and 300,000 immigrants per year - half the amount which has arrived annually in recent years - if low-skilled workers are included." The report noted, "The shortage of highly qualified professionals in the technology sector, especially in the Internet area, as well as health professionals, engineers and consultants is starting to become urgent." The study cited "Spain's low birth rate and aging population...as reasons for the continued need for immigration." The findings come as the Spanish government plans "to slash the number of jobs on offer to foreigners recruited in their countries of origin, mostly in low-skilled areas like construction and the services sector." It also "reduced the total number of professions requiring foreign workers by 35 percent."

Problem 6: Predict the population which Spain will have in July 2016 by using the migration model (5) and the following data: Throughout the eight years from 2008 to 2016 Spain will have h = 250,000 new immigrants per year, as suggested by the survey quoted above. For the first four years until 2012 use the current birth and death rates for Spain provided on indexmundi.com. However, if a large part of the immigrants will be qualified workers, in particular in the health professions, then Spain's death rate will decrease. Thus assume that for the period from 2012 to 2016 the value of b in (5) has been reduced by 10 percent. Also, as highly educated parents tend to have smaller numbers of children, assume that the birth rate a is reduced by 5 percent.

Hint: Problem 6 can be done by using solution tool for (5) in two steps: First, work with the current population, birth and death rates to find Spain's population four years from now. Read off this population as best you can from the graph and use it as the new value for P_0. Also find the modified values of a and b to calculate the population change over the next four years.

Project 2: Electrical LRC Series Circuits

The charge *q(t)* as a function of *t* in an electrical LRC series circuit, see Figure 2 below, satisfies the second order linear differential equation

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E(t). \tag{1}$$

Here *C* is the capacitance, measured in farads *F*, *R* is the resistance, measured in ohms Ω, and *L* the inductance, measured in henrys *h*. The electromotive force *E(t)* is measured in volts *V*. The charge *q(t)* will always be expressed in coulombs *C*, while the current $i(t) = q'(t)$ is measured in amperes *A*.

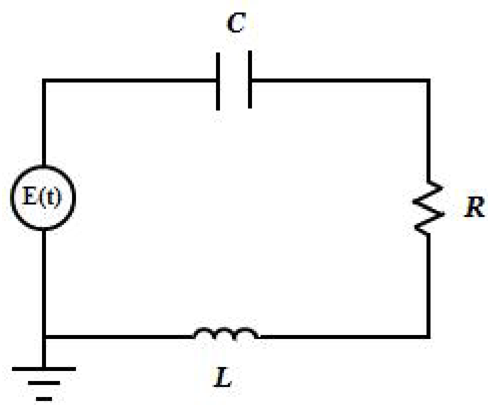

Figure 1: The LRC series circuit with electromotive force $E(t)$.

The differential equation (1) is very similar to the differential equation

$$mx''(t) + \beta x'(t) + kx(t) = f(t). \tag{2}$$

for a spring-mass system with driving force *f(t)*. Therefore all the phenomena that appear in oscillating mechanical systems do also arise in electrical LRC circuits and the underlying mathematical methods are the same. The main goal of this project is to understand this in concrete examples.

A guiding theme is that we will study how capacitors get charged or discharged in different circuits. Thus you will be asked to include plots of the charge function *q(t)* for the various circuits considered. To do this use the tool "**project2tools.nbp**", which is uploaded together with this assignment in the homework folder on the course web site. It allows you to plot all functions of the type

$$q(t) = e^{-\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t)) + c_3\cos(\gamma t) + c_4\sin(\gamma t) + c_5,$$

where all of the variables α, β, γ and c_1 to c_5 can be entered (many of these parameters will be zero in the concrete examples). Also, the domain and range can be adjusted. In each plot you should pick domain and range in a way which best brings out the interesting features of the function. Note that the time scales for charging or discharging capacitors are typically just fractions of seconds.

Free Electrical Oscillations

Throughout this section we will assume that there is no electromotive force, i.e. E(t) = 0 in (1). Thus we will study free electrical oscillations within an LRC series circuit. Specifically, we will study how an initially charged capacitor gets discharged.

Problem 1: Let us first assume that the circuit has no resistance, i.e. $R = 0$. This is an idealization that is not possible in real circuits (where even the wires alone will cause a small amount of resistance). Also assume that the inductance is 8 mh and the capacitance is 2 mF (where mh and mF denote milli-henry and milli-farad, i.e 1 mh = 10^{-3} h and 1 mF = 10^{-3} F).
(a) If the initial charge on the capacitor is 4 C and no current is flowing, find the charge *q(t)*.
(b) Find the current *i(t)* as well as its frequency and period. Include a plot of *q(t)*.
(c) What is the amplitude of the current, i.e. its largest possible magnitude? Find the first time when this happens.

Problem 2: Assume that a 0.2 Ω resistor is added to the circuit of Problem 1.
(a) Find *q(t)* and *i(t)* under the same initial conditions as in Problem 1(a). Also plot *q(t)*.
(b) Find the frequency and the period of the current *i(t)*.
(c) Find the first time *t* at which the capacitor is completely discharged.
(d) Comment on the effect of adding resistance to a circuit, i.e. the changes in amplitude, frequency and period of the current between Problem 1 and Problem 2.

Forced Electrical Oscillations

We will now study forced electrical oscillations, meaning that there is a non-zero electromotive force *E(t)*. We will first look at an example with a DC-source (direct current), where $E(t) = E_0$ is constant in time. Then we will study an AC electromotive force (alternating current) given by a sinusoidal function $E(t) = E_0 \sin(\gamma t)$.

Problem 3: Add a DC-electromotive force of *E(t)* = 10 V to the LRC-circuit from Problem 2.
(a) Find the charge *q(t)* and current *i(t)* under the assumption that the initial charge and initial current both vanish. Plot *q(t)*.
(b) Also find the steady-state charge and steady-state current after a long time *t*.

Problem 4: Consider the LRC-circuit with an AC impressed voltage of *E(t)* = 10 sin(100πt) V, which corresponds to a 50 hertz frequency with a transformed amplitude (10 rather than 110 volts). Again assume that the initial charge and current vanish.
(a) Find *q(t)* and *i(t)* by solving the differential equation (1) using the method of undetermined coefficients. Plot *q(t)*.
(b) Find the steady-state charge and steady-state current after a long time as a function of *t*.
(c) In Problems 3 and 4 the electromotive force has the same amplitude, once as a DC source and once as an AC source. Compare the amplitudes of the steady-state charges in both problems. Try to explain this.

For a general LRC-series circuit with AC source

$$E(t) = E_0 \sin(\gamma t). \tag{3}$$

the *reactance X* and *impedance Z* are defined by

$$X = L\gamma - \frac{1}{C\gamma}, \qquad Z = \sqrt{X^2 + R^2}. \tag{4}$$

Both are measured in ohms.

It can be shown (and is not part of the project) that the steady-state current of an LRC-series circuit with source *E(t)* given by (3) is given by the formula

$$i_p(t) = \frac{E_0}{Z}\left(\frac{R}{Z}\sin(\gamma t) - \frac{X}{Z}\cos(\gamma t)\right). \tag{5}$$

You may use (5) to check your result in Problem 4(b), which is a special case.

Problem 5:

(a) Find a formula for the amplitude of the steady-state current (5).

(b) Suppose the values of *L*, *R*, *C* and E_0 are given as in Problems 1 to 4. How would you have to choose the forcing frequency γ in (3) in order to get the largest possible amplitude of the steady-state current? How can this be written in terms of the reactance *X*? How big is the amplitude of the steady-state current in this case?

A Circuit with Two Loops

The LRC-series circuit in Figure 1 is very simple to analyze because it contains only one loop. The differential equation (1) is found by using Kirchhoff's second law: We add up the voltage drops $Lq''(t) = Li'(t)$ at the inductor, $Rq'(t) = Ri(t)$ at the resistor and $\frac{1}{C}q(t)$ at the capacitor, and set them equal to the electromotive force $E(t)$.

In the "real" world (and in your engineering circuits course) electrical circuits are generally much more complicated. Modern integrated circuits often contain hundreds of loops, with many interconnected inductors, resistors, capacitors and other electronic devices. Mathematically, circuits with multiple loops are described by systems of linear differential equations. This is a topic in advanced differential equations that is not covered in EGR 265. But we will be able to analyze one relatively simple circuit with two loops that is given in Figure 2. For this we will have to use both Kirchhoff laws.

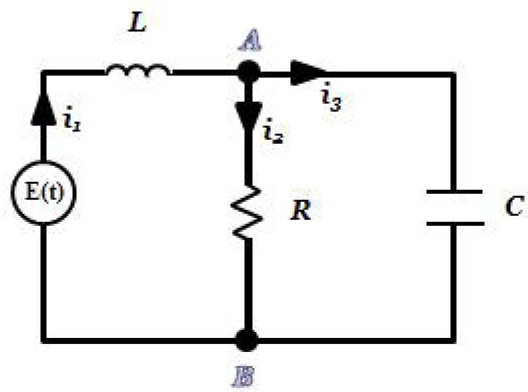

Figure 2: The LRC series circuit with electromotive force *E(t)*.

The current $i_1(t)$ coming from the source $E(t)$ splits at the point A into two currents $i_2(t)$ and $i_3(t)$, which are joined together at B again into $i_1(t)$.
The First Kirchhoff law says that

$$i_1(t) = i_2(t) + i_3(t). \tag{6}$$

We now have to use Kirchhoff's second law separately on each of the two loops. In the left loop we have a voltage drop $Li_1'(t)$ at the inductor and a voltage drop of $Ri_2(t)$ at the resistor. Note here that we have to use different currents at the inductor and the resistor. This is set equal to the source $E(t)$, leading to the equation

$$Li_1'(t) + Ri_2(t) = E(t). \tag{7}$$

The right loop has a voltage drop of $\frac{1}{C}q(t)$ at the capacitor, where $q(t)$ is the charge at the capacitor which is related to the current in the right loop by $i_3(t) = q'(t)$. We also have a voltage drop $Ri_2(t)$ at the resistor, but this has to be counted with a negative sign (think of going through the right loop clockwise, then you pass the resistor against the current $i_2(t)$). There is no source term in the right loop. Thus Kirchhoff's second law gives for the right loop that

$$\frac{1}{C}q(t) - Ri_2(t) = 0. \tag{8}$$

Equations (6), (7) and (8) give a full mathematical description of the circuit in Figure 2.

Problem 6: Consider the circuit in Figure 2 with the same inductor, resistor and capacitor as in Problems 1 to 5 and switch on a DC electromotive force E(t) = 10 volts at time t = 0. Assume that initially no current is flowing and that the capacitor is uncharged. Find explicit functions for all the currents $i_1(t)$, $i_2(t)$ and $i_3(t)$and for the charge $q(t)$ at the capacitor. Also, what charge builds up at the capacitor after a long time?

The solution of Problem 6 is quite involved. You should do this step-by-step as described in the following:

Step 1: By differentiating the equation (8) and using that $i_3(t) = q'(t)$ one gets $\frac{1}{C} i_3(t) - Ri_2'(t) = 0$. If we also use the first Kirchhoff law (6) we get

$$i_1(t) - i_2(t) - Ri_2'(t) = 0. \tag{9}$$

Equations (7) and (9) together form a typical system of *two coupled first order differential equations* for the two functions $i_1(t)$ and $i_2(t)$. Being coupled means that both of them contain $i_1(t)$ and $i_2(t)$. Thus we can not solve them with the standard methods which we learned for first order DEs: To get $i_1(t)$ by solving (7) we would have to know $i_2(t)$ first, to get $i_2(t)$ from (9) we would have to know $i_1(t)$ first. Learning how to get out of this vicious circle is the main content of the theory of *systems of differential equations*. However, in our example we can avoid having to use the general theory and continue with the following steps.

Step 2: Differentiate the DE (9) one more time and solve the resulting equation for $i_1'(t)$. Substitute this for $i_1'(t)$ in (7). The result can be rewritten as

$$LRCi_2''(t) + Li_2'(t) + Ri_2(t) = E(t). \tag{10}$$

Note that (10) is a second order linear DE for $i_2(t)$. Thus we have managed to transform a system of two coupled first order DEs into a single second order DE! In order to solve (10) we need to know the initial conditions $i_2(0)$ and $i_2'(0)$. The problem provides $i_2(0)$ and one can find $i_2'(0)$ by using (9).

Step 3: Find $i_2(t)$ from the initial value problem for (10) by standard methods.

Step 4: The previous equations provide relations that allow us to find $i_1(t)$ and $i_3(t)$ once $i_2(t)$ is known.

Step 5: Finally, use the relation between $i_3(t)$ and $q(t)$ to find $q(t)$, also using that the capacitor is initially uncharged.

Problem 7: Plot $q(t)$. Compare the resulting charge function with the one in the circuit in Problem 3, which was built from the same inductor, resistor and capacitor and hooked up to the same DC electromotive force. What are the changes in how the capacitor is charged? Compare the limiting charges as well as the decay rates and frequencies of the transient parts of the charge.

Project 3: Designing a Dumpster

Extreme values of functions of two variables
We start by recalling the **Closed Interval Method** from single-variable Calculus (see for example Stewart, Essential Calculus, Section 4.1):

To find the absolute maximum and minimum of a continuous function f on a closed interval [a, b]:

1. *Find the values of f at the critical numbers of f in [a, b] (i.e. the numbers c such that $f'(c)$ = 0 or the derivative doesn't exist),*

2. *Find the values of f at the endpoints of the interval,*

3. *The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.*

Here we want to consider an extension of the Closed Interval Method to functions of two variables $f(x, y)$ whose domain is a subset D of $\mathbb{R}^2$.

Just as a closed interval contains both of its endpoints, we say that a subset *D* of $\mathbb{R}^2$ is closed if it contains all its boundary points. Examples of closed sets are the disk

$$\{(x;\ y) |\ x^2 + y^2 \leq 1\}$$

or the square

$$\{(x, y) |\ 0 \leq\ x\ \leq 1, 0\ \leq y\ \leq 1\},$$

while the triangular region

$$\{(x, y) |\ x > 0, y > 0;\ x + y < 1\}$$

is not closed.

Below we will also say that a subset of $\mathbb{R}^2$ is **bounded** if it fits into a finite disk.
We will only consider functions $f(x, y)$ which have gradients $\nabla f(x, y)$ for all points in their domain. For such a function we say that *(a, b)* is a **critical point** if

$$\nabla f(a, b) = 0,$$

i.e. both partial derivatives f_x and f_y are 0 at (a, b). This means that the tangent plane to the graph of f at (a, b) is horizontal.

We can now state the following generalization of the Closed Interval Method:

To find the absolute maximum and minimum values of a differentiable function f of two variables on a closed, bounded set D in $\mathbb{R}^2$:

1. *Find the values of f at the critical points of f in D.*
2. *Find the extreme values of f on the boundary of D.*
3. *The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.*

Example: Find the absolute maximum value and the absolute minimum value of $f(x, y) = 2x^2 + xy + y^2 - y$ on the triangular region D given by the set of all (x, y) such that $x \geq -1, y \leq 1$ and $y \geq x$, see Figure 1.

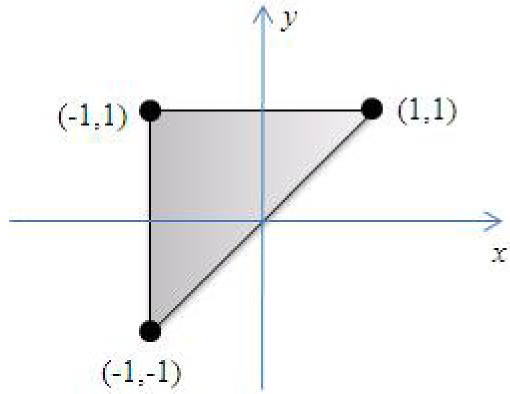

Figure 1: The domain $D = \{(x, y) |\ x \geq -1, y \leq 1, y \geq x\}$

We start by finding the critical points of f: The partial derivatives are

$$f_x = 4x + y, \qquad f_y = x + 2y - 1.$$

With algebra we find that the only point (x_0, y_0) such that $4x_0 + y_0 = 0$ and $x_0 + 2y_0 - 1 = 0$ is given by $(x_0, y_0) = \left(-\frac{1}{7}, \frac{4}{7}\right)$. This is the only critical point at which we have

$$f\left(-\frac{1}{7}, \frac{4}{7}\right) = -\frac{2}{7}.$$

Now we have to find the largest and smallest values of f on the boundary of D. This boundary consists of three pieces:

(i) the horizontal line segment $y = 1, -1 \leq x \leq 1$,
(ii) the vertical line segment $x = -1, -1 \leq y \leq 1$, and
(iii) the diagonal line segment $y = x$ for $-1 \leq x \leq 1$.

On each of these pieces we can use the usual Closed Interval Method to find extreme values:

(i) On the line $y = 1$ the function $f(x, y)$ becomes $g(x) = f(x, 1) = 2x^2 + x$, where $-1 \leq x \leq 1$. We have $g'(x) = 4x + 1 = 0$ if $x = -1/4$. This gives the value $g\left(-\frac{1}{4}\right) = f\left(-\frac{1}{4}, 1\right) = -1/8$.

(ii) On the line $x = -1$ the function $f(x, y)$ becomes $h(y) = f(-1, y) = y^2 - 2y + 2, -1 \leq y \leq 1$. This has the critical number $y = 1$, where $h(1) = f(-1, 1) = 1$.

(iii) On the diagonal $y = x$ the function $f(x, y)$ becomes $k(x) = f(x, x) = 4x^2 - x, -1 \leq x \leq 1$. The critical number is $x = 1/8$ and $k\left(\frac{1}{8}\right) = f\left(\frac{1}{8}, \frac{1}{8}\right) = -1/16$.

Finally, we have to consider the values of f at the endpoints of the three line segments, meaning the three corners of the triangle: Here we have $f(1, 1) = 3$, $f(-1, -1) = 5$ and $f(-1, 1) = 1$ (where the last one already appeared as a critical number in (ii)).

Of all the numbers at which we evaluated f, the largest value was

$$f(-1, -1) = 5.$$

This is the absolute maximum value of f on D. The absolute minimum value of f on D is at the critical point of ∇f:

$$f(-1/7, 4/7) = -2/7.$$

Designing a dumpster as a mathematical optimization problem

The strategy described above can be used in applications in the form of optimization problems. This is similar to the optimization problems that are studied in single variable calculus. But we are now able to solve more complex optimization problems, which lead to having to determine extreme values of functions of several variables.

We will consider the problem of minimizing the cost of building a dumpster. We will do this in several different ways, considering an increasing number of contributions to the total cost of the dumpster. This is an example of a problem in optimal design. We will use three different mathematical models to describe the cost function. The more sophisticated models should give a better description of the underlying real world problem. However, we will see that the price for this is that the mathematics involved in finding the optimal result becomes more and more complicated. In fact, calculus methods (like the ones described in Section 2 above) will not be sufficient to handle the more complicated models. We will also use numerical methods, here in the form of computer graphing tools. This is quite typical in mathematical modeling. Real world problems are usually so complex that we need all the help we can get to understand them mathematically: analytical methods from algebra and calculus, graphical and numerical methods.

Minimizing the cost of steel

Your company has received an order of multiple steel dumpsters, which need to have a rectangular base and a volume of 200 cubic feet. Denote the dumpster's width in feet (from left to right) by x and its depth in feet (from front to back) by y. Its height at the front is z feet, while it is required to be one foot higher at the back to let rain water flow off the lid, see Figure 2.

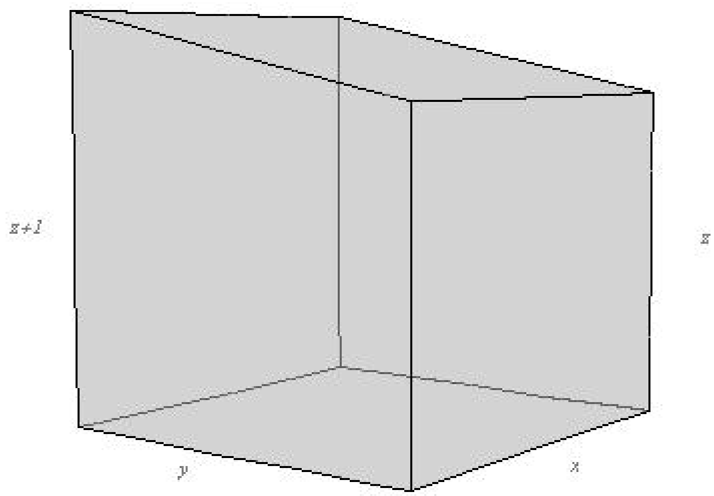

Figure 2: A no-frills dumpster

Problem 1: The two sides, front and back of the dumpster are to be made of 12-gauge steel, which according to the US Standard Gauge for Stainless Steel is 0.1094 inches thick and costs \$0.90 per square foot. The base is made of 10-gauge steel, which is 0.1406 inches thick and costs \$1.20 per square foot. The lid costs \$70, regardless of shape. We also have the following constraints on the dimensions of the dumpster: For security reasons, the front edge must be at least 4 feet high (to prevent small children from climbing into the dumpster). Also, the dumpster should be at least 5 feet wide and deep to accommodate large trash items (like calculus books, etc).

> (a) Within the given constraints, find the dimensions x, y, z of the dumpster which minimize the cost of steel. Here, and in all later problems, provide these dimensions up the the closest 1/8 of an inch, which is the highest precision your sheet steel cutting machine allows to specify.
> (b) Could you build an even cheaper dumpster of the required volume if there were no constraints (and you wouldn't mind your kids playing in the trash)? If yes, provide the dimensions.

Use the following step-by-step instructions to solve Problem 1:

Step 1: Express the cost of steel required for building a dumpster as a function $C(x, y, z)$.

Step 2: Use the fact that the volume of the dumpster must be 200 ft^3 to eliminate the variable z from the expression for $C(x, y, z)$. Find the resulting function of two variables and call it $c(x, y)$.

Step 3: Use the constraints on x, y and z to describe the domain D of the function $c(x, y)$ in R^2. Sketch D and also describe it by inequalities for x and y.

Note: You immediately get two inequalities for x and y. The height restriction on the front of the dumpster together with the volume formula also leads to an inequality relating the size of x and y. You should end up with a domain D in $\mathbb{R}^2$ whose boundary is given by three lines and/or curves.

Step 4: Use the method described in Section 2 to find the absolute minimum value of the function $c(x, y)$ on the domain D, which is the answer to Problem 1(a). Be sure to do all of the following to find the absolute minimum value of $c(x, y)$. Find its values at all critical points in the interior of D. Separately, find the minimal values of c on all three boundary curves. Also, check the values of c at the three "corners" of D.

Step 5: From the information found so far you can also answer Problem 1(b). Make sure to provide the optimal values of x, y and z in each case.

Minimizing the cost of steel and paint
In the remaining two problems, for simplicity, we will not consider any constraints on the dimensions of the dumpster. Thus, the main task will be to identify critical points of the resulting two-variable functions.

Problem 2(a): The local city council has recently decided that your town will compete in the state beautification contest. Therefore all dumpsters need to be painted in pink. The lid comes pre-painted and the dumpsters are to be placed against walls. Thus only the front and two sides need to be painted. The paint costs $0.20 per square foot. Again, the volume of the dumpster is to be 200 cubic feet. Which dimensions (x_0, y_0, z_0) minimize the total cost of steel and paint?
You may follow a similar strategy as in Problem 1, finding a cost function $C(x, y, z)$ and then, after eliminating z, $c(x, y)$. However, it will not be possible to find the critical points of $c(x, y)$ by only using calculus and algebra methods. Proceed as follows: Find both partial derivatives and set them equal to zero, $c_x = 0$, $c_y = 0$. The equation $c_y = 0$ can be solved for x. Do this and insert the resulting expression for x into the equation $c_x = 0$. You will be able to write the resulting equation in the form

$$ay^4 + by + c = 0. \tag{1}$$

This is a fourth-order equation, whose roots are hard to find by algebra methods (there are known formulas for this, but they are quite complicated and rarely used). Instead, use the first of the two graphing tools provided in

project3tools.nbp

to plot the function on the left hand side of (1), using the values you found for a, b and c and appropriate choices for domain and range.

You will find that the function has two positive roots. Use Newton's Method (find this method in a book or on the internet if you have forgotten about it) to calculate both roots up to a precision of at least two decimals, where you can read off initial approximations from the graph. For each of the two values for y which you get, find the corresponding values of x and z. If your numbers are correct, you will find that only one of the resulting points (x, y, z) makes sense for our application. Call this point (x_0, y_0, z_0). This is the answer to Problem 2(a) after being re-written in terms of feet and eighths of inches.

Problem 2(b): Include a plot of the function $ay^4 + by + c$ in your project.

(c) We concluded above that (x_0, y_0, z_0) had to be the solution to our problem, because the problem should have a solution and it was the only reasonable one we found with our method. However, mathematically the critical number (x_0, y_0) does not necessarily have to be a minimum of the function $c(x, y)$, it could also be a maximum or a saddle point. Use the second graphing tool in **project3tools.nbp** to plot the function $c(x, y)$.

This tool allows you to plot 3D graphs of functions of the type

$$c(x,y) = c_1 xy + \frac{c_2}{xy} + c_3 x + \frac{c_4}{x} + c_5 y + \frac{c_6}{y} + c_7,$$

where all constants $c_1, \ldots, c_7$ may be chosen. The x and y domain can be chosen as intervals $[x_{min}, x_{max}]$ and $[y_{min}, y_{max}]$. Use a good xy-domain and a good perspective (by working with the 3D-rotation feature of the tool) to get a plot which clearly shows that (x_0, y_0) is the minimum of the function $c(x, y)$. Attach a print-out of this plot to your report. Hints: You should avoid letting x or y get close to 0 because the graph gets very steep there, which is not what we are interested in. It may also help to plot more than one view of the graph.

Minimizing the cost of steel, paint, welding and hinges

Problem 3(a): Fifteen minutes before your workers start to cut the sheet steel you realize that there will also be costs for welding and for attaching the lid to the base of the dumpster, which might affect the optimal dimensions. The cost for welding the four sides and base along their common edges is \$0.25 per foot. The lid is attached to the base at its back edge with a series of hinges that cost \$0.75 per foot. Find the dimensions (x_0, y_0, z_0) that minimize the total cost of steel, paint, welding and hinges. In fifteen minutes!

After you have set up the new cost functions $C(x, y, z)$ and $c(x, y)$ you may try to solve this problem by again using the method from Section 2, i.e. by finding the gradient of $c(x, y)$ and determining its critical points. However, you would find that the algebra becomes so difficult that this is practically impossible.

Instead, use the 3D graphing tool in project3tools.nbp to solve this problem graphically. Start with a choice of $[x_{min}, x_{max}]$ and $[y_{min}, y_{max}]$ which you are reasonably certain to contain the minimizer (x_0, y_0) of $c(x, y)$. Rotating the 3D graph to view it horizontally when looking from the direction of the y-axis you should get a better idea of where x_0 is lying. In the same way, looking from the x-direction you get an estimate for y_0. Based on this you can modify $[x_{min}, x_{max}]$ and $[y_{min}, y_{max}]$, choosing smaller intervals to get a zoomed version of the graph. Continue this until you can read off x_0 and y_0 correct up to two decimals. Find the corresponding z_0 and change units to feet and eighths of inches to get your answer.

Problem 3(b): Include two plots of the final zoom-level of the graph of $c(x, y)$, viewing the graph horizontally from the x- and the y-direction.

Qualitative Assessment of the Projects

While a quantitative assessment of the Engineering Mathematics course versus the Calculus III and Differential Equations courses is needed and forthcoming, qualitatively students appear to have a very positive reaction to the projects and their use in forging understanding of various mathematics topics covered in the course. Students successful in passing the course have graduated in engineering and matriculated into medical, dental, law, and graduate schools as well as the workforce. The faculty member in Mechanical Engineering teaching the Fluid Mechanics course mentioned that he has seen no problem in the students taking the Engineering Mathematics course, stating that they are every bit as capable and committed as the students from the traditional track, though a more formal study is warranted.

If students voting with their money are any indication of the success of the course, the course has grown substantially over time, with two sections needed in the fall semester to handle the rather large numbers. During the initial offering in the fall of 2008, there was one section with 50 students completing the course. Last fall, there were 69 students completing the course in one section and 49 in the other. Even the spring section has grown to the initial fall 2008 levels as there are currently 54 students in the course, up from the 29 students completing the course in the first spring offering of 2009. I would like to reiterate that these numbers reflect only those students who have completed the course, since there are a handful of students who drop the course every semester.

For direct student comments, all of the IDEA Survey comments have been gathered, beginning with the initial offering of the course in fall 2008. Below are <u>all</u> of the comments with the keyword project. When asked "Was the balance of theory and applications right for the course (both instructors)? Explain." the students gave the following comments (again, only comments relating to the projects are shown, though the course receives high marks [>4.5/5.0] as well):

- Yes, the projects helped with theory and applications learned from the notes and book.
- Yes, projects were given to allow for testing us on applications and tests were given to test theory.
- Theory and applications were not overdone although some of the applications were difficult to get through.
- There should be more applications. Personally, I don't think the projects are adequate if the goal of the class is for students to utilize the theory and threading together the concepts and putting them to applications. There should be more hands-on applications. The class should also be more frequent than twice a week.
- Stolz seemed to do more theory, and Moore did more applications. It all seemed to work well for the class. We were tested fairly on what was covered in class. The projects are a little much, with the homework, tests, and 4 other classes to prepare for.
- Yes, it was well balanced. The projects were very difficult and I didn't like them, but when it came down to it I realized I learned most of what I was doing from the projects. It might even be a good idea to put the projects before the test so that the material would really be hit home. I probably would have improved on my test grades if I had done the projects first.
- EGR/Math 265 is a great course. It has taught me a lot about applying math to engineering. I learned more in this math class than any other math course I have ever

taken. It was challenging, but no too hard where you feel like giving up either. It is a busy class, but I think it will help me in the future. Dr. Stolz and Dr. Moore were very good professors and really cared about us learning and applying what we knew. Even though, I kept doing average in the class, I still worked hard and wanted to learn what they were teaching. I think the school of engineering should require that all engineering students take this instead of Calculus III.

- I have to say the projects, although stressful, were incredibly helpful. They were worthwhile, teamwork-based assignments that helped me understand the material, as well as improve my formal scientific and engineering writing skills.
- Yes, the course was a perfect blend of lecture, projects, and independent work with instructor help.
- The projects help students to think critically and doing them is a good practice in applying the concepts in class.
- Overall I believe that this was an excellent class in preparing engineering students to solve engineering problems with mathematical concepts. There was a great balance of providing the theory behind these methods, and the application of these methods to real world scenarios. The projects and homeworks both provided ample opportunity in practicing these methods in preparation for the tests. I would recommend for anybody thinking about taking this class to take it with Dr. Nkashama and Dr. Moore. They work really well together and really do a great job with the class.
- Yes I believe that there is a good mixture of learning styles. The projects played a huge part in my learning. I liked getting the basic math concepts and then learning how to apply them to my field.
- I like the importance of the projects, I am not the greatest test-taker and the projects let me display the knowledge I have gained.

Summary

A four-hour course at the University of Alabama at Birmingham has been created to increase the level of engineering and scientific discourse in mathematics. Course notes have been developed and refined through the years, providing exact material used in the course. Projects have also been developed to align with the topics of the course in order to give students a bit of perspective on pertinent problems. Students are encouraged to reasonably work together with the sole requirement that each report is written individually – bringing forth the meaning and awareness of the individual student. The course has been well-received by the manner in which we draw students.

Acknowledgement

I must acknowledge the tireless efforts of my colleague Dr. Günter Stolz, Professor in the Department of Mathematics at the University of Alabama at Birmingham, who co-created the course, the notes for the course, and each of the projects. Dr. Günter Stolz also team-teaches the course often, giving the course far more perspective than I ever could have alone. I have also taught the course and gained perspective from other colleagues in the Department of Mathematics at UAB, namely Dr. Mubenga Nkashama, Dr. Junfang Li, and Dr. Yanni Zeng. The collaboration between the School of Engineering and the Department of Mathematics has

been wonderful. I look forward to working with my colleagues in the Department of Mathematics every semester. Dr. Rudi Weikard, Chair of the Department of Mathematics, has been wonderfully supportive as well, rotating in a new faculty member every so often.

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