AC 2012-3176: USING PROJECTS TO STIMULATE LEARNING IN MATHEMATICS AND ENGINEERING MATHEMATICS COURSES

Dr. Hassan Moore, University of Alabama, Birmingham

Using projects to stimulate learning in mathematics and engineering mathematics courses

Abstract

An engineering mathematics course – developed in the fall of 2008 at the University of Alabama at Birmingham – teaches concepts in both Calculus III and Differential Equations. An important aspect of the course is the implementation of student projects, meant for individual performance, which challenge students to apply the information taught through modeling a system, analyzing it, and presenting a solution(s) complete with units and an interpretation of the physical phenomena examined. Given that time is a premium in the academy and project creation is a time consumer, three original projects, which may be injected into any Calculus III, Differential Equations, or Engineering Mathematics course, have been created. The aim is to increase the use of projects in courses where the intent is there but creation time is the mitigating factor. This paper includes projects that tackle first-order ordinary differential equations (ODEs), second-order ODEs, and multivariable calculus.

Introduction

In some mathematics courses, students are asked to perform a number of homework sets, tested, and then given a cumulative examination at the end of the course to examine proficiency. There have been many calls for and attempts at mathematics pedagogy reform[1-4]. The physical nature of what students learn may never be called to task since mathematicians (fairly) are there to teach the principles of mathematics – onto which physical understanding may be built later in science and engineering courses. But, students may form deeper connections with the material when they contend with it in context[5-10], bridging physics and engineering into the mathematics course. Projects[11-15] provide a great mechanism for students to investigate different physical phenomena, connecting it with current mathematics structures taught.

At the University of Alabama at Birmingham, an engineering mathematics course has incorporated the use of projects to stimulate students as they delve into an extended problem that relates contextually to some physical problem and increases their awareness of dimensional analysis and units. It provides an extended bridge into the types of problems that they may one day solve, especially since homework problems are not representative of the projects many in engineering may face in the work force. Projects may narrow the disconnect between the short, segmented units by relating current content to former courses, new ideas, and current issues in society. With little to no insight on the projects from the professors, who team teach the course, students should be able to complete the project with the material taught in class.

Below are the primary course topics covered:

I. First-Order Ordinary Differential Equations (ODEs)
   A. Basic Concepts, Modeling
   B. Initial Value Problems
   C. Direction Fields
   D. Existence and Uniqueness
   E. Separable ODEs

F. Linear ODEs
G. Applications

II. Second-Order ODEs
A. Homogeneous Linear ODEs with constant coefficients
B. Free Oscillations
C. Forced Oscillations
D. Electrical/Mechanical Systems

III. Multivariable Calculus
A. Functions of Several Variables
B. Partial Derivatives, Gradients, Directional Derivatives
C. Line Integrals
D. Multiple Integrals
E. Spatial Transformations, Center of Mass, Moments

Each of the projects include a cover page, which along with the title and due date provide instructions reminding students of the course expectations, including acceptable collaboration as well as the structure of the report. These projects may be made more simple or difficult as needed. Every project begins with the following instructions (below is an example of the instructions given for a project on first-order ordinary differential equations):

> The project described below is self-contained, meaning that you should be able to do it by carefully reading through it and using what you learned in class about first-order ordinary differential equations.
>
> A carefully written report is expected, which can be done in (legible) handwriting or typed with a text processor. You do not need to copy the problems into your report, but should clearly label to which problems your answers refer. Include the calculations that lead to your answers. Wherever appropriate, in particular if you are asked to state and justify an opinion, write your answers in full sentences and adequate English. Whenever numerical answers are required, find the exact values using a calculator.
>
> A certain amount of collaboration is acceptable in doing this project, but **reports must be written individually**. Thus, when writing your report, make sure that it is clearly different from reports of others and reflects your own thoughts for solving the problems. Reports that are virtually identical to others will not receive credit.

In project 1, students are asked to use their knowledge of solving first-order ordinary differential equations to tackle basic ideas in rocket science through the examination of the Jules Verne catapult idea and single-stage rockets. The second project explores ideas in mechanical vibrations with the concept of resonance. The third project extends project 1 by examining the fuel distribution in two-stage and three-stage rockets.

Project 1: Rocket Science for Starters

Projectiles without Air-Resistance

The goal of this project is to use first-order ordinary differential equations based on Newton's law to study some simple models of rocket flight. In particular, we will only study rockets that remain close to the surface of the Earth. This allows us to assume that the force on the rocket due to gravity is given by mg, where m is the mass of the rocket and $g=9.8$ m/sec^2. In one model we will take air resistance into account, assuming constant air density which also holds only near the Earth's surface.

A warning about notation: We use the same abbreviation for mass and meter. To distinguish between the two, we use an italicized m for mass and regular font m for meters. Make sure to not mix them up in your work.

Using many simplifications, we will not come anywhere near the complicated models and equations which NASA is using. But our simple models will already give some interesting insights in how rockets need to be designed.

Our first model is not really a model of a rocket, but we will rather study a projectile that is shot up vertically into the air. The science fiction writer Jules Verne imagined in the 19th century that rockets could be built like this, with no engines, reaching space by being shot up with high initial velocity at take-off. We will try to get some insight into the feasibility of this idea.

We will denote the height above ground of the rocket by $x(t)$, with the surface of the Earth giving the origin. Thus the positive x-axis is oriented upwards, see Figure 1. As usual, we write $v(t) = x'(t)$ and $a(t) = x''(t) = v'(t)$ for velocity and acceleration.

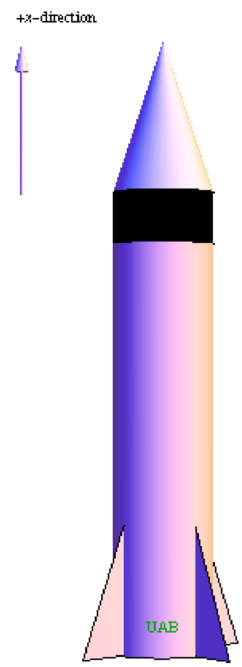


Figure 1: UAB goes to the Moon

In the first problem the projectile is shot up vertically without taking air resistance into account, meaning we consider a basic *vertical throw*. Of course, there are well known physics formulas for this, but you are asked to not use these formulas and instead rediscover them by solving DEs. After being shot, the only force acting on the projectile is gravity $-mg$, with the negative sign indicating that gravity acts downwards. Thus Newton's Law gives $ma(t) = -mg$. With $a(t)=v'(t)$ and using $v(0)=v_0$ as initial condition this gives a first order initial value problem for the velocity:

$$mv'(t) = -mg,\ v(0) = v_0. \tag{1}$$

Problem 1:

(a) Solve the IVP (1) to find an expression for $v(t)$ in terms of v_0, g and t. (There is no m-dependence here, as the mass cancels out in (1).)

(b) The height above ground of the projectile satisfies the IVP

$$x'(t) = v(t),\ x(0) = 0. \tag{2}$$

where we take into account that the projectile is launched from ground level. Insert the result from (a) for $v(t)$ and solve the IVP (2) to find an expression for $x(t)$ in terms of v_0, g and t.

(c) The projectile reaches its maximal height at the time t_{max} when $v(t_{max})=0$. Find t_{max} and $x(t_{max})$.

(d) What initial velocity v_0 is required for the projectile to reach a height of 10 km? Give the answer in m/sec as well as in miles per hour, to get a better sense of the required speed.

Projectiles with Air-Resistance

Let us now explore how air resistance influences a vertical throw. We will assume that the force on the projectile due to air resistance is proportional to its velocity $v(t)$, with a constant of proportionality k of unit *kg/sec*. As air resistance causes a force opposite to the velocity, this term also appears with a negative sign in Newton's Law. Thus the IVP (1) now takes the modified form

$$mv'(t) = -mg - kv(t),\ v(0) = v_0. \tag{3}$$

As opposed to Problem 1, the mass m does not cancel out any more.

Problem 2:

(a) Solve the IVP (3) as a separable DE to get an expression for $v(t)$ in terms of v_0, g, t, k and m.

(b) From $v(t_{max})=0$ find an expression for the time t_{max} at which the projectile reaches its maximal height.

(c) Find an expression for $x(t)$ by solving (2), this time using the answer to Problem 2(a) for $v(t)$.

(d) Find an expression for $x(t_{max})$. Try to simplify.

(e) Insert the explicit values $m=10$ kg, $k = 0.03$ kg/sec, and $g=9.8$ m/sec^2 into the expression for $x(t_{max})$. It then becomes a function of v_0 alone. Use a good graphing program (for example MATLAB, Maple or Mathematica) to produce a plot of this function. From the plot, read off the smallest value of v_0 so that the projectile is

guaranteed to reach a height of 10 km. You may want to “zoom” your plot, meaning to do multiple plots using smaller and smaller domains until you can read off v_0 to, say, at least one decimal. Print out the plot(s) and include it in your report. (Note that the equation $x(t_{max}) = 10000$ is too complicated to be solved for v_0. At best you could use Newton's method to find its root.)
(f) Compare the results of Problems 1 and 2 and comment on the changes caused by air resistance.

Suppose the projectile is launched through a *100*-meter long vertical canon, in which the projectile is accelerated with a constant acceleration a. Let v_0 denote the velocity at which the projectile leaves the canon. By an argument similar to the one used in solving Problem 1(c) it is seen that $100 = v_0^2 / 2a$.

Problem 3: How much would the acceleration a during launch through the canon have to be in order to reach the launch velocities required in Problems 1 and 2? Express your answer in multiples of g (for example, $5g$ for an acceleration of 49 m/sec^2). Find some information about the maximal g-forces that humans can tolerate (and give a reference). Comment on the chances for realizing Jules Verne's ideas in human space flight.

Remark: A calculation that we will not do here shows that it takes a launch velocity of approximately 11 km/sec for a projectile to escape the Earth's field of gravity. This is even higher than what you should have found above. It takes the correct gravity force at high altitude into account (given by Newton's Law of Gravity, not just mg), but ignores air resistance (which doesn't play a role at high altitude).

One-Stage Rockets
After having given up on the concept of blasting a rocket into space with a canon, we will now do what rocket scientists had to do anyway, namely use booster rockets. These have an engine that burns off fuel and expels the resulting gas to generate a thrust force. If α is the amount of fuel burned per second, measured in kg/sec, and β is the velocity at which the gas is expelled, in m/sec, then the thrust is given by $\alpha\beta$ (in Newtons). This force acts upward. Also including gravity (near the surface of Earth) and ignoring air resistance (to simplify the equation), we get from Newton's Law that

$$mv'(t) = \alpha\beta - mg.$$

This looks easy to solve, but there is one problem: By burning fuel the rocket loses mass! This happens at a rate of α kg/sec, so the mass left after t seconds is $m = m_0 - \alpha t$, where m_0 is the mass of the rocket at launch (including fuel). Inserting this into Newton's equation and using the idea that a booster rocket sits on its launch pad with initial velocity zero, we find the initial value problem

$$(m_0 - \alpha t)\, v'(t) = \alpha\beta - g(m_0 - \alpha t),\ v(0) = 0. \qquad (4)$$

Fortunately, this is still a first order separable DE for the velocity.

Problem 4:

(a) Solve the IVP (4) to get an expression for $v(t)$ in terms of m_0, g, t, α and β.

(b) If m_r is the initial mass of the non-fueled rocket and m_f the amount of fuel which is loaded, then $m_0 = m_r + m_f$. Burning fuel at a rate of α kg/sec, it takes a time of $t_{max} = m_f / \alpha$ to burn all the fuel. This is the time of burnout of the booster. By inserting this into the result of (a), find an expression for $v(t_{max})$ in terms of m_r, m_f, α, β and g. Try to simplify.

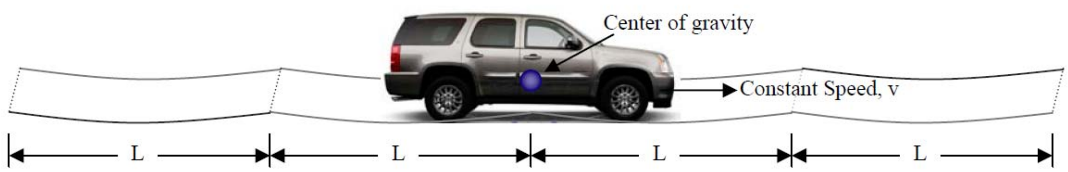
(c) Solve the following optimization problem: Which amount of fuel should be loaded for the rocket to reach the maximum velocity at burn-out? Proceed as follows: In the expression for $v(t_{max})$ found in (b), replace m_f by the variable x and think of the result as a function of x. Now use Calculus methods to find the value of x where the function has its maximum. Call this value x_{max} and write down a formula for x_{max} in terms of α, β, m_r and g.

(d) *(Bonus question) Can you explain the answer to (c) directly from differential equation (4) without solving it?

We plan to explore multi-stage rockets later in the semester. For this we will first need to learn some multi-variable calculus methods.

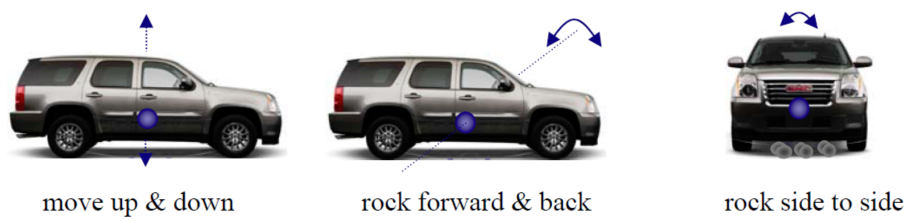
Background to Project 2: A Model of Resonance for Car Suspensions

Consider a car traveling along the road that consists of poured concrete sections of equal length. Due to seasonal variability (expansions / contractions), moisture, and variable forces consistently applied (weight of vehicles), concrete sections may bow. If you have ever driven on a bank of bowed slab road with a uniform length, L, traveling at constant speed, v, you may have experienced what is called resonance. But, let us explore further.



It takes (L/v) seconds to travel a distance L at a speed v (recall: distance = rate * time). Then, the vehicle hits a "ridge" in the road at a frequency of (v/L). At some speed, v, this frequency can excite the vehicle so that it bounces up and down. Thus, the roadway acts as a forcing.

Note that depending on the roadway and the average frequency (v/L), the car could



or any combination of these.

Generally, the mode of vibration that is excited at the lowest (v/L) is associated with greatest motion (largest displacement amplitude). This is called the fundamental mode of vibration. In many engineering applications, the fundamental mode is the only one examined as it is the most problematic.

If you can understand the behavior of the system (system here, is the car and its suspension system) vibrating in its fundamental mode, you can build on this understanding to analyze all possible modes of vibration.

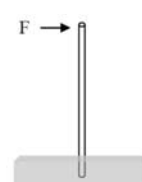
Undamped, vibrating systems

Now, vibrating systems are characterized by

- Inertia – mass that has to be acted on by a force to get it moving and which has kinetic energy when moving, and
- Stiffness – a resistance to displacement.

Consider pushing a vertical pole that is firmly mounted in the ground. When a force is applied to the pole (see figure), the pole will deflect a small distance, δ. For small δ, you can represent the relationship between F and δ as

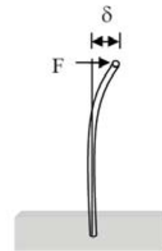
$$F = k \cdot \delta$$



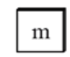
Here, k represents the <u>stiffness</u> of the pole and is measured in units of either N/m or lb_f/ft.

Notes regarding stiffness:

- Given the same force, F, a stiff pole (large k) will not deflect as far.
- The structure “resists” the displacement.
- When the applied force is taken away, the structure moves back towards its original position (and is likely to oscillate until it settles back into its equilibrium position).



Now, the inertia of the system for the fundamental mode of vibration is modeled as a lumped mass that moves in only one direction (up and down for the car and side to side for the pole).

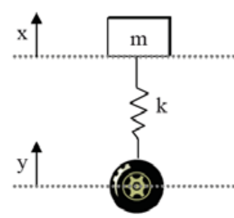


The stiffness is represented by a spring, which is schematically drawn as

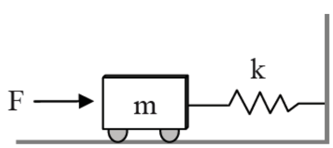
When the motion is up and down

When the motion is side to side

So, the fundamental vibration mode for the car can be modeled by the figure to the right. In the figure, *y* represents the displacement input at the center of the wheel and *x* is the deflection of the lumped mass.

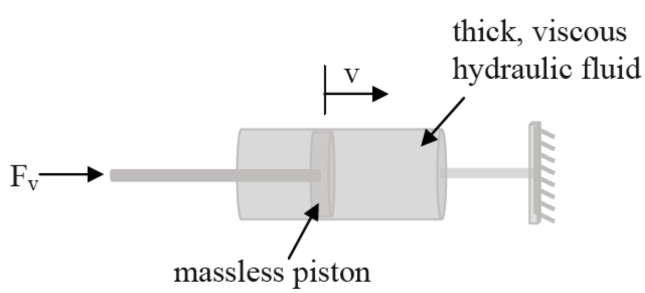


The fundamental vibration mode of the pole could be modeled by



where *F* is the force causing the displacement. The wheels are assumed to have frictionless bearings and roll without slip.

These models are excellent “first cuts”, but if a car starts moving up and down or a pole starts oscillating side to side and the input stops, the vibration will eventually die out due to frictional losses. This loss mechanism that dissipates energy is modeled as

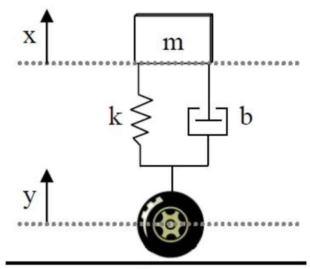


where $F_v \propto v$ or $F_v = bv$.

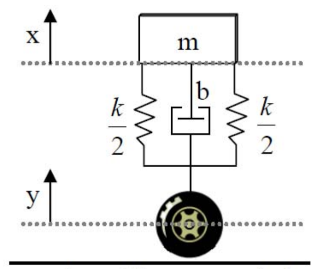
This idealization of the dissipation in the system yields in a friction force that is proportional to the velocity of the mass and results in a linear ordinary differential equation for the system. Because of the ease of obtaining a solution and the power of the resulting solution, we often linearize problems in engineering.

Models with damping

The damped single degree of freedom (SDOF) model for the car is indicated as shown below. The schematic on the right shows the stiffness split into two springs, so as to suggest a 'balanced' system.

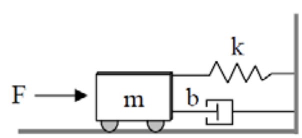


suspension system

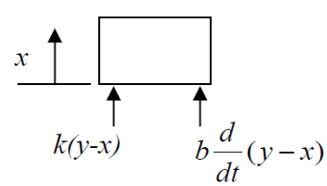


suspension with apparent balance

The schematic of the model for the oscillating pole, when damped is accounted for, is shown at right. Here, too, the stiffness could and sometimes is split into two springs, one on either side of the damper.



Mathematical Model for the Car:



$$\sum F = m\ddot{x}$$

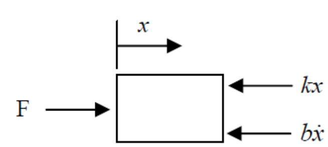
$$m\ddot{x} = k(y-x) + b\frac{d}{dt}(y-x)$$

$$m\ddot{x} = k(y-x) + b(\dot{y}-\dot{x})$$

$$m\ddot{x} + b\dot{x} + kx = b\dot{y} + ky$$

But, we start first with $m\ddot{x} + kx = ky$.

Mathematical Model for the Pole:



$$m\ddot{x} = -kx - b\dot{x} + F$$

The minus signs represent the fact that both stiffness and damping will resist the motion.

Project 2: A Model of Resonance for Car Suspensions

A Mathematical Model for the forces on a car suspension system

In class a mathematical model for the forces on the suspension system of a car traveling on a road that consists of poured concrete sections was introduced. It resulted in the second order linear differential equation

$$m\ddot{x} + b\dot{x} + kx = b\dot{y} + ky \qquad (1)$$

We use dot-notation for the first and second time-derivatives of the functions *x=x(t)* and *y=y(t)*, as common in physics and engineering.

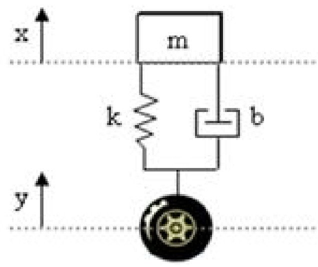


Figure 1: Single degree of freedom model of a car

Here we lump the mass *m* of the car into a single point at the center of mass of the car, whose vertical deflection from its equilibrium position is described by the function *x(t)*. For the car traveling on an uneven road, the function *y(t)* represents the elevation of the road under the car at time *t*, which can also be viewed as the vertical displacement of the wheels. The constant *k* is the spring constant representing the stiffness of the springs. For our model we can look at one spring of stiffness *k* replacing four springs (one at each wheel) of stiffness *k*/4. The constant *b* describes the viscosity of the fluid used in the hydraulic suspension units.

We will use a cosine function to model the profile of the concrete sections of the road. Thus, using the *z*-axis for the direction of the road, the elevation of the road is given by

$$h(z) = a\cos\left(\frac{2\pi z}{L}\right). \qquad (2)$$

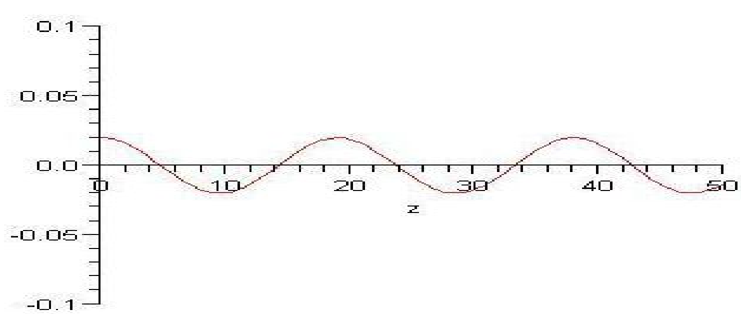


Figure 2: The function *h(z)* for *a=0.02* m and *L=19* m

Note that the period L of $h(z)$ represents the length of the sections, while $2a$, the maximal change of elevation, measures how much the sections are distorted. The axes in Figure 2, where the profile of the road is given for $a=2$ cm and $L=19$ m, are not to scale (the amplitude is much smaller relative to the length of the sections).

Problem 1: The car travels at constant velocity v in the z-direction. Find its position $z(t)$ after time t (assuming $z(0)=0$) and insert this into (2) to find the function $y=y(t)=h(z(t))$ in (1). Also find $\dot{y}$ and insert both into (1). What is the resulting differential equation?

Resonance for an undamped suspension system

The result of Problem 1 is an inhomogeneous second order linear differential equation with constant coefficients. We will consider the special case where $b=0$, i.e. there are no hydraulic units in the suspension, or the driver hasn't noticed that the hydraulic liquid has leaked out. In this case the DE found in Problem 1 describes driven motion without damping. When solving this DE we will work with the initial values $x(0)=0$ and $\dot{x}(0)=0$, i.e. the car initially rests in its equilibrium position.

This initial value problem will have to be solved in part (c) of Problem 2 below, separately for the resonant and non-resonant case. Use the methods studied in class to do this, in particular the methods of characteristic equations and undetermined coefficients. You should observe that the guess required in the method of undetermined coefficients will be different for the resonant and non-resonant case.

Problem 2:

(a) Express the frequency $\omega/2\pi$ of free vibrations and the frequency $\gamma/2\pi$ of the driving force in terms of m, k, L and v.

(b) Express the condition for pure resonance in terms of an equation for m, k, L and v.

(c) Find an explicit formula (depending on m, k, L and v) for the solution $x(t)$ of the DE found in Problem 1 with $b=0$ and initial conditions $x(0)=0$ and $\dot{x}(0)=0$. Do this separately for the resonant and non-resonant cases. Use the abbreviations ω and γ from part (a) as well as $F_0 = ka/m$ to simplify the formulas.

So far our general considerations did not require a choice of units. All further work should be done in the mks-system, using meters (m) for distance, kilograms (kg) for mass and seconds (s) for time. In particular, this means that the unit for k becomes kg/s^2 (as made necessary by the fact that the terms $m\ddot{x}$ and kx in (1) must have the same unit).

One can measure the spring constant k for a given spring as follows: If an additional mass m_0 adds (or subtracts) a length s_0 to the elongation of the spring, then $k=m_0g/s_0$ ($g=9.8$ m/s^2). Here one should not use masses which are too large to avoid overstretching or overcompressing the spring.

Problem 3: Measure the cumulative spring constant of the suspension system of your or a friend's car by an experiment: How much does the body of the car go down if a person of given mass enters it? What is the corresponding value of k in kg/s^2? The person should try to stay between the two front seats, near the center of mass of the car, to distribute his/her weight evenly over the four wheels.

Problem 4: Assume that the length of the concrete sections of the road is 19 meters. Find out the mass of your car in kilograms. Using the value of the cumulative spring constant found in Problem 3 and assuming that all the hydraulic liquid of your car's suspension has leaked out, calculate the critical velocity v_{crit} at which your car gets into the state of pure resonance when you drive the car. Express your answer in miles per hour to get a better sense of this velocity.

Problem 5: Keep assuming that the concrete sections are 19 meters long, but work with a standardized car of mass 1250 kg and cumulative spring constant 120000 kg/s^2.

(a) Assuming your mass is 80 kg, find v_{crit} for this car.
(b) What is the effect of putting two additional linebackers into the rear seats of the car on v_{crit}?
(c) At what velocity does a series of speed bumps at 4 meter spacings get the car into resonance, still assuming broken hydraulic units (and that you kicked out the linebackers)? Comment on what is achieved by putting speed bumps in a residential neighborhood.
(d) Plot the solution curve for $x(t)$ where L, m, k and $v = v_{crit}$ are as in part (a). Use $a = 2\ cm$, i.e. the street level varies by 4 cm within 19 meters. (Note: For plotting in this problem and in Problem 6 below use any plotting software you are familiar with or, alternatively, the online plotting instructions which will be provided separately.)
(e) Assume that the bottom of your car is 10 inches above the ground when the car is at rest. If you drive at resonance speed, how long does it take for the bottom of the car to hit the road?

Problem 6: Suppose you drive at a speed v which is 5% lower than the resonance speed from Problem 5(a). Plot the solution of the initial value problem $x(0)=0$, $\dot{x}(0) = 0$ for the DE found in Problem 1 for this v. Comment! How would this ride end?

Problem 7: A car manufacturer wants to lower production costs by constructing a car suspension system without hydraulic units. The spring should be stiff enough to make sure that cars of a weight of up to 3000 kg can drive at speeds up to 80 mph without getting into resonance. How stiff should the springs be, expressed in the choice of the cumulative spring constant k? How much bigger is this in percent compared to the value of $k = 120000\ kg/s^2$ which we used above? How do you imagine it would feel to drive this car?

Project 3: Multi-Stage Rockets

Summary of Results on One-Stage-Rockets

In the project "Rocket Science for Starters" we used a differential equation based on Newton's Law to find the velocity of a one-stage booster rocket:

$$(m_0 - \alpha t)\, v'(t) = \alpha\beta - g(m_0 - \alpha t). \tag{1}$$

This uses the following notation:

$v(t)$ = velocity measured in m/sec
m_0 = initial mass of the rocket (including fuel) measured in kg
α = kilograms of fuel burned by the rocket engine per second
β = velocity of the expelled gas in m/sec
$g = 9.8$ m/sec^2

We had solved this first order DE assuming that the initial velocity is zero, but it is easily seen from our methods that the solution of (1) under the more general initial condition

$$v(0) = v_0. \tag{2}$$

is

$$v(t) = -\beta \ln(m_o - \alpha t) - gt + \beta \ln(m_0) + v_0,\ 0 \le t \le T_{out}. \tag{3}$$

By T_{out} we denote the time of burnout of the rocket. If m_f is the total amount of fuel loaded onto the rocket and α is the amount of fuel burned per second, then $T_{out} = m_f / \alpha$.

Two-Stage Rockets

Now consider a rocket consisting of two stages plus a payload of mass m_p on top, see Figure 1.

We make several assumptions:

- Both stages have the same rocket engine, meaning the same values for α (fuel burned per second) and β (velocity of expelled gas).
- The total amount of fuel available for both stages has mass m_f (in kilograms). We are free to choose how much fuel we put on each stage. Let x be the amount of fuel on the first stage and y be the amount on the second stage. Thus

$$x + y = m_f.$$

- Assume that each stage (weighed without fuel) has the same mass as the amount of fuel loaded on it. Thus the total mass (rocket plus fuel) of the first stage at takeoff is $2x$ kilograms, while the total mass of the second stage is $2y$ kilograms.

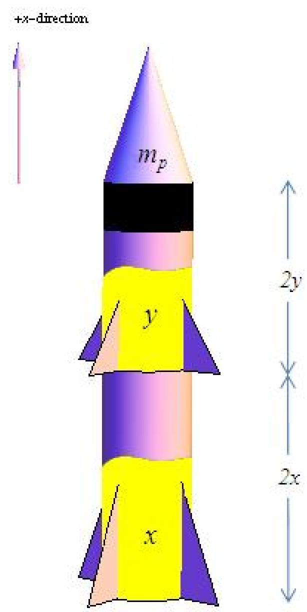


Figure 1: A two-stage rocket

The last assumption is based on the idea that it takes a bigger stage to hold a larger amount of fuel. That the mass of the stage and the fuel are exactly the same is probably not very realistic, but it will keep our equations relatively simple while still leading to reasonable results.

The flight of the rocket takes place as follows:

- At the time of ignition of the first stage the initial velocity of the rocket is zero.
- The first stage burns at full engine strength until all fuel is used up. At the time of burnout the first stage is ejected.
- The second stage is ignited at the time of burnout of the first stage and burns at full strength until the fuel is used up.

The ultimate goal of the following series of problems is to find the velocity of the rocket at the time of burnout of the second stage and then to optimize how the fuel should be distributed among the two stages: How should we choose *x* and *y* to reach the highest possible final velocity of the rocket?

As the size of each stage will be proportional to the amount of fuel it carries, the answer to this question will give us an idea on how multi-stage rockets are shaped, in particular, how the different stages relate to each other in size. Before we start our calculations, you might want to look at some real life rockets in Figure 2 to get a sense of what to expect.

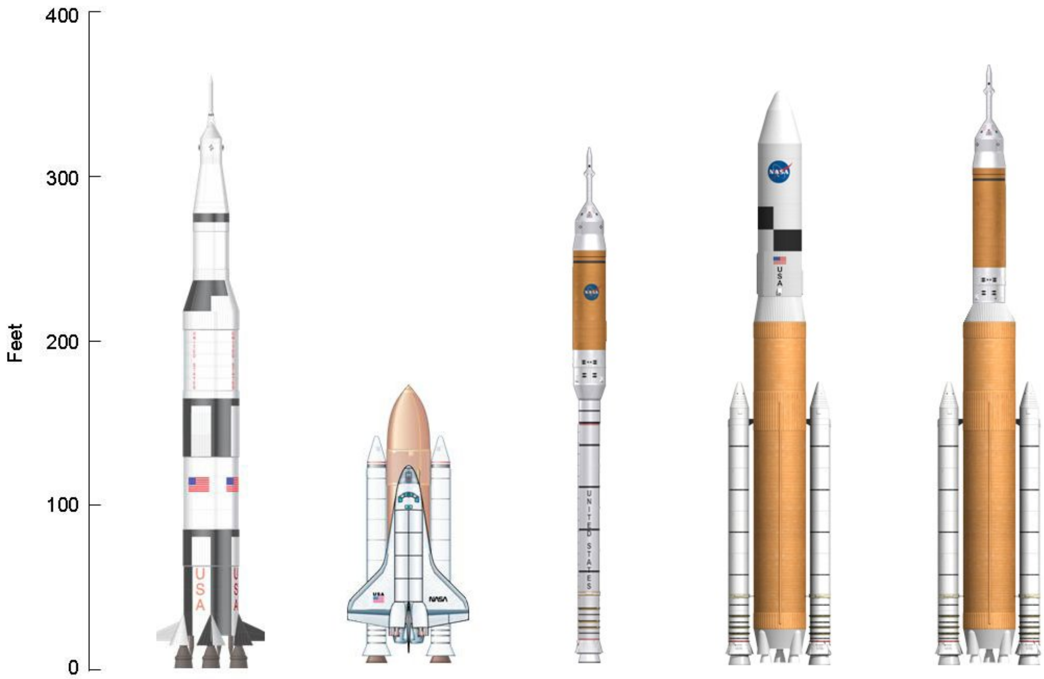


Figure 2: NASA rockets: Saturn V, Space shuttle, Ares I, Ares V, Ares IV

Now let's get serious. We start by studying the velocity of the rocket during the burn of the first stage.

Problem 1: Let $T_{1,out}$ be the burnout time of the first stage and $v_{1,out}$ the velocity of the rocket at burnout of the first stage. Find a formula for $v_{1,out}$ in terms of m_p, x, y, α, β and g.

Note: This problem is solved by using (3) with initial condition $v_0 = 0$ and the correct values for m_0 and $t = T_{1,out}$. In this case m_0 is the total mass of the rocket at takeoff, i.e. payload plus fuel plus the mass of both stages (which can be expressed in terms of m_p, x and y). $T_{1,out}$ should be expressed in terms of α and x.

Next we consider how the velocity changes while the second stage is burned.

Problem 2:

(a) Let $T_{2,out}$ be the burn-out time of the second stage (where we reset the time to zero at the time of ignition of the second stage). Find the velocity $v_{2,out}$ of the rocket at time $T_{2,out}$ in terms of m_p, x, y, α, β and g.

(b) Use the fact that $x + y = m_f$ to eliminate y and express $v_{2,out}$ in terms of m_p, m_f, x, α, β and g. Assuming that m_p, m_f, α, β and g are fixed constants, denote the resulting function as $f(x)$.

Note: Part (a) can be done by again using (3), this time with initial velocity $v_0 = v_{1,out}$ from Problem 1 and initial mass m_0 given by the payload and the total mass of the second stage. Also, express $T_{2,out}$ in terms of α and y.

Problem 3: Fix $m_f = 20$ kg, but keep m_p variable. Find the value of x between 0 and m_f which maximizes $f(x)$. Call this value x_0.

Note: Problem 3 is solved by using calculus methods to find the maximum value of the function $f(x)$ in the interval $0 \leq x \leq 20$. For the derivative of $f(x)$ you will find a sum of several fractions. To find the roots of $f'(x)$, determine a common denominator, so you only need to find the roots of the numerator. After some tedious algebra this numerator will turn out to be a quadratic function of x, with two roots. It can be seen that one root is larger than 20, which cannot be a solution of our problem. Thus the other root will be the correct value for x_0. This will turn out to be a function of m_p (but not depend on α, β and g).

Hint: To simplify the numerator, after having found the common denominator, try entering it into the command field at wolframalpha.com.

Problem 4: Explore how x_0 depends on m_p by plotting it as a function of m_p. Use different domains for m_p to answer the following questions (e.g. for small and large payload, respectively).

(a) The optimal amount of fuel in the second stage is $y_0 = 20 - x_0$. Thus the initial mass of the first stage is $2x_0$, the mass of the second stage is $2y_0$. Which stage is larger? Does this depend on the payload m_p?

(b) What happens to the size of the first stage relative to the size of the second stage if the payload is very small (say only a few grams)?

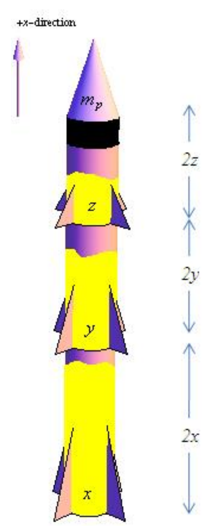
(c) What happens if the payload is very large (like several hundred kilos)?

Include plots to justify your answers.

A Three-Stage Rocket

The method that we used in the previous section to determine the velocity of a two-stage rocket, i.e. using the formula (3) separately for each stage, can also be applied to rockets with more than two stages. Of course, the mathematics will get increasingly complicated and we will soon depend on computer help to do it.

Let us consider a three-stage rocket, see Figure 3, where we will make the values of all the quantities involved completely explicit.



A three-stage rocket

We will assume that the payload is 10 kg and that for each of the three stages we have $\alpha = 0.2$ kg/sec and $\beta = 4000$ m/sec. The total amount of fuel to be distributed over the three stages has mass $m_f = 30$ kg, and we denote by x, y and z the mass of fuel on the first, second and third stage, respectively. Thus we have $x + y + z = 30$. As in the previous section we assume that the total mass of each stage (with fuel) is twice higher than the mass of the fuel alone, i.e. the stages have mass $2x$, $2y$ and $2z$.

We will not make you go through the process of finding the velocity of each of the three stages and instead provide the final result. The velocity of the rocket at the time of burnout of the third stage, after eliminating z through the formula $z = 30 - x - y$, is given by

$$v_{3,out} = -150g + 4000[-\ln(40 - x - y) + \ln(70 - 2x - 2y) - \ln(70 - 2x - y) + \ln(70 - 2x) - \ln(70 - x) + \ln(70)]. \qquad (4)$$

This is a function of two variables x and y. The domain D of this function is the triangular region in $\mathbb{R}^2$ described by $x \geq 0$, $y \geq 0$, $x + y \leq 30$, see Figure 4, reflecting the fact that at most 30 kg of fuel can be put on the first two stages, leaving $z = 30 - x - y$ kg for the third stage.

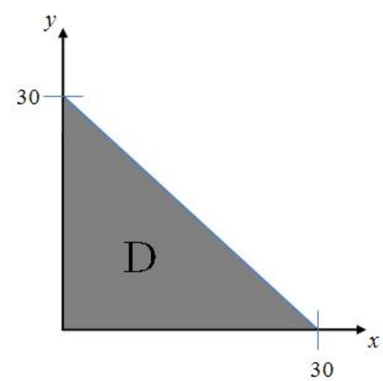


Figure 4: The domain in Problem 5

Problem 5: Use the plotting capabilities at wolframalpha.com to numerically find the coordinates (x_0, y_0) of the point in the domain

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 30\}$$

where $v_{3,out}$ takes its maximal value. Determine x_0 and y_0 correct to at least one decimal and find $z_0 = 30 - x_0 - y_0$, the optimal amount of fuel on the third stage. Attach a plot with sufficient magnification to show the correct values of x_0 and y_0. Comment on the relative size of x_0, y_0 and z_0 and thus of the three stages. In particular, look at the ratios x_0 / y_0 and y_0 / z_0. What do you notice?

Note: Instead of plotting the function on the right hand side of (4) you can simplify the function somewhat by observing that constant terms and constant positive factors do not influence where the maximum of the function lies. Thus you can instead find the maximum of the function

$$f(x,y) = -\ln(40 - x - y) + \ln(70 - 2x - 2y) - \ln(70 - 2x - y) + \ln(70 - 2x) - \ln(70 - x).$$

For example, on wolframalpha.com the command

$$\text{plot } \sin(x + y),\ x = 0..5,\ y = -3..3$$

will produce a contour plot and a 3D plot of the function sin$(x + y)$ over the domain $0 \leq x \leq 5$, $-3 \leq y \leq 3$. For the function $f(x,y)$ from above you should start with plotting it on the square domain $0 \leq x \leq 30$, $0 \leq y \leq 30$, only looking for maximal values in the triangle D. The contour plots are particularly useful in identifying high lying plateaus of the function. You can assume that the maximal value lies near the center of such plateaus and use this to repeatedly zoom the domain until you can read off the coordinates of the maximum with the required accuracy.

Summary

A mathematics-based course that follows Calculus I and Calculus II with an engineering problem-solving emphasis was developed based upon engineering faculty interviews. Each major topical area includes a significant engineering challenge with governing equations from physics as the mathematical techniques are taught. Units and reasonableness of solutions are included as vital skills. This course is taught using active, collaborative, and problem-based learning approaches.

Acknowledgement

I must acknowledge the work of Dr. Gunter Stolz, Professor in the Department of Mathematics at the University of Alabama at Birmingham, who co-created the notes for this course and each of the projects. Also, Dr. Sally Anne McInerny, Professor and Head, Mechanical Engineering at the University of Louisiana at Lafayette, College of Engineering, was very instrumental in the brain-storming and creation of the prelude to the Second Order ODE project on Mechanical Systems. Team-teaching and the collaboration involved in this course have provided various insights, creating a wonderful course which receives high marks from students every semester.

Bibliography

1. Ruane, M. (2001). Calculus Reform, Differential Equations and Engineering. *Proceedings of the 2001 ASEE Annual Conference & Exposition*.
2. Foertsch, J., Moses, G., Strikwerda, J., & Litzkow, M. (2002). Reversing the Lecture/Homework Paradigm Using eTEACH® Web-based Streaming Video Software. *Journal of Engineering Education*, 91(4), 267.
3. Barr, R.E., Pandy, M.G., Petrosino, A.J., Roselli, R.J., Brophy, S., & Freeman, R.A. (2007). Challenge-Based Instruction: The VaNTH Biomechanics Learning Modules. *Advances in Engineering Education*, 1.
4. Parsons, J.R., (2002). Engage Program: Implementing and assessing a new first year experience at the University of Tennessee, *Journal of Engineering Education*.
5. Pendergrass, N.A., Kowalczyk, R.E., Dowd, J.P., Laolache, R.N., Nelles, W., Golen, J.A., & Fowler, E. (2001). Improving First-Year Engineering Education, *Journal of Engineering Education*, 33.
6. Bass, H., (1997). Mathematicians as Educators, *American Mathematics Society*, 44(1), 18.
7. Prince, M.J. & Felder, R.M., (2006). Inductive Teaching and Learning Methods: Definitions, Comparisons, and Research Bases. *Journal of Engineering Education*, 123.
8. Prince, M. (2004). Does Active Learning Work? A Review of the Research. *Journal of Engineering Education*, 223.
9. Litzinger, T.A., Lattuca, L.R., Hadgraft, R.G., & Newstetter, W.C. (2011). Engineering Education and the Development of Expertise. *Journal of Engineering Education*, 100(1), 123.
10. Froyd, J.E. and Ohland, M.W. (2005). Integrated Engineering Curricula. *Journal of Engineering Education*, 147.
11. Ulseth, R.R., Froyd, J.E., Litzinger, T.A., Ewert, D., & Johnson, B.M. (2011). A New Model of Project Based Learning in Engineering Education, *Proceedings of the 2011 ASEE Annual Conference & Exposition*.
12. Padir, T., Muller, K.O., & Coullard, C., (2008). Teaching Differential Equations in a Diverse Classroom. *Proceedings of the 2008 ASEE Annual Conference & Exposition*.

13. Pennell, S., Avitabile, P., & White, J., (2006). Teaching Differential Equations with an Engineering Focus. *Proceedings of the 2006 ASEE Annual Conference & Exposition*.
14. Munasinghe, R.A., (2005). Multidisciplinary Research Projects for Engineering Students. *Proceedings of the 2005 ASEE Annual Conference & Exposition*.
15. Mohammadzadeh, A. (2004). Learning from a Golf Ball. *Proceedings of the 2004 ASEE Annual Conference & Exposition*.