Volleyball: A New Twist On Terminal Velocity

Thomas W. Cairns, Ph. D.

Department of Mathematical Sciences

The University of Tulsa

Abstract

The terminal velocity problem is a popular classroom example for Calculus instructors. It is often posed in a version that is simplified to the point of being unrealistic. This paper uses a volleyball as the example object and computes terminal velocities under a variety of assumptions. Actual values for the drag coefficient are used and computations are done in Mathematica. The computations illustrate how the drag crisis intervenes and complicates the mathematical analysis. The *Mathematica* code used is provided in an appendix. No attempt has been made to assess the efficacy of this approach in the calculus classroom.

Introduction

Terminal velocity problems have long been popular with instructors in calculus and differential equations.[1] They can be posed in their full aerodynamic complexity or simplified in a variety of ways that meet the needs of the moment and exemplify the mathematical ideas that are being studied. They are inherently one-dimensional and so can be posed that way with no loss of generality.

The idea is that an object is dropped from a high altitude, accelerated by gravity and this acceleration is opposed by an aerodynamic force called *drag*. Later we'll be specific by making the object a volleyball. Gravity acts downward and drag acts in the direction opposite to the direction of motion. So all forces are vertical and a single axis is required to represent the position of the object at any later time. We'll assume here that upward is the positive direction. Drag is in some sense proportional to the square of the object's speed. One would expect that the object would be accelerated by gravity with rapidly increasing opposition from drag and would asymptotically approach a velocity that could be computed by setting the drag equal to the weight of the ball.

So how does this work out in practice? We would need to determine the velocity at time from what we know about the aerodynamics of the object. Questions might arise also about position and acceleration but the main questions of interest here have to do with velocity. All computations below are done in *Mathematica* and the required code is given in the appendix.

Zero drag case

The simplest case, included in every calculus text, is the one in which the drag is assumed to be zero which means that the ball is dropping in a vacuum. In this case one begins with Newton's second law, **f** = m**a**, where ||**f**|| is the weight of the ball (direction down), m is its mass and **a** is a the acceleration due to gravity. Since gravity accelerates every object downward at about $9.8\, m/s^2$, this simplifies to $\mathbf{a} = -9.8\, \boldsymbol{j}\, m/s^2$ where **j** is the vertical unit vector. The problem is one dimensional and so the mathematics can play out in its scalar version $a = -9.8\, m/s^2$.

Now integrate this expression with respect to t. The units are dropped below but, of course, t is in seconds and v and v_0 are m/s. v_0 is ± the initial speed. If the problem is proposed in its most general form, v_0 will be positive or negative as the direction is up or down.

$$\int a\, dt = -\int 9.8\, dt$$
$$v = -9.8\, t + v_0$$

Integrate again wit respect to t. y_0 is the initial position. The units for y and y_0 are meters.

$$\int v\, dt = -\int 9.8\, t\, dt + \int v_0\, dt$$
$$y = -4.9\, t^2 + v_0\, t + y_0$$

A variety of problems can be posed at this point. They might involve assuming that the ball was not just dropped with an initial velocity of 0, but was propelled up or down. But the problem we're considering here is the terminal velocity and these generalizations provide no additional insight. So we'll assume that $v_0 = 0$ and $v = -9.8\, t$. This is too simple a case to be really interesting because it never happens that the drag is zero and, if it did, the limiting velocity would work out to be $\lim_{t \to \infty} v = -\infty$.

Real world cases

In the interest of realism we're dropping a volleyball which is an object we know a lot about. Common sense tells us that the effect of drag on a moving object and its terminal velocity as well depend heavily on its shape and construction, especially the nature of its surface that is presented to the airstream. Sports balls offer an interesting and useful choice. In recent decades there have been many studies on the drag and lift of sports objects, especially balls, moving through air.[2,3] In this case lift usually is a result of spin. The technical name for lift due to spin is the *Magnus effect*. Good examples include a topspin forehand drive in tennis and an overhand curveball from a baseball pitcher. Examples of sports objects that lift by aerodynamic effects other than Magnus effect are frisbees, javelins and ski jumpers. The balls we drop mathematically to study terminal velocity will not be spinning. If they were, a lateral force would be produced by the Magnus effect and the ball would be diverted from a vertical path. So sports balls it is and, in particular, volleyballs. Why volleyballs? We'll see that volleyballs occupy an interesting spot among the family of sports balls with respect to aerodynamic effects.

So let's get to the real mathematics starting with the general case and paring it down to our special application. The equation of motion of an object moving in air is $m \frac{dv}{dt} = m\boldsymbol{g} - \frac{1}{2} \rho A C_D v^2 \boldsymbol{\tau} + \frac{1}{2} \rho A C_L v^2 (\boldsymbol{\sigma} \times \boldsymbol{\tau})$.[4] This comes from setting the sum of the forces to zero. Commonly used values for the entries in this equation specific to a volleyball are below.

m – mass = .268 kg

g – acceleration due to gravity = $-9.8\, m/s^2$

ρ – mass density of air = $1.23\, \text{kg}/m^3$

A – cross sectional area = $\pi \left(\frac{.214}{2}\right)^2 m^2 \doteq .036\, m^2$

$\boldsymbol{\tau}$ – unit tangent vector in the direction of motion

$\boldsymbol{\sigma}$ – unit normal vector

$\boldsymbol{\sigma} \times \boldsymbol{\tau}$ – unit binormal vector

C_D – drag coefficient

C_L – lift coefficient

Because of the assumption of no spin, $C_L = 0$ reduces the equation to

$$m \frac{dv}{dt} = m\boldsymbol{g} - \frac{1}{2} \rho A C_D v^2 \boldsymbol{\tau}$$
$$\frac{dv}{dt} = -\frac{1}{2m} \rho A C_D v^2 \boldsymbol{\tau} + \boldsymbol{g}$$

where the magnitude of the drag is $.5\rho A C_D v^2$ and so is a function of the square of the speed. Now introduce y(t) as the position so that v'(t) = y''(t) and evaluate $.5\,\rho A/m = .0825$. Then the one dimensional equation becomes

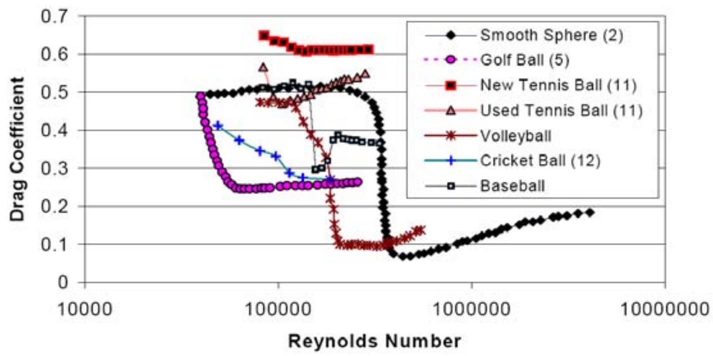
$$y'' = -.0825 C_D |y'|\, y' - 9.8$$

Even if C_D were constant, it would seem futile to search for an analytic solution to this differential equation. As it is, C_D is known only from experimental data.

Drag coefficient for a volleyball

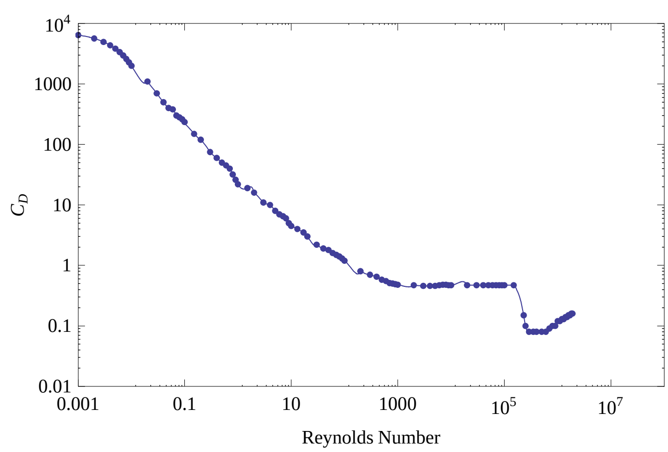
To determine C_D experimentally Beatrice Hahn and David McCulloch, two students at The University of Michigan, did a wind tunnel study in 1999 to determine the drag on a non-spinning volleyball as a senior project in aeronautical engineering under the direction of Dr. Don Geister.[5] The results were an excellent set of data that were included along with other balls in a paper by Dr. Rabi Mehta of NASA Ames and Dr. Jani Pallis of Cislunar Inc and shown in Fig. 1.[2]

Figure 1. Drag Coefficients for Sports Balls



These data are available in numerical form in the Appendix and are used for C_D in this paper. Fig. 2 contains a plot of the points. The curve through the data is the *Mathematica* interpolation thereof.

Figure 2. C_D for a Volleyball



The graph above is a log-log plot and the units for the horizontal axis are Reynolds numbers. Drag coefficient graphs are universally displayed in this way. The Reynolds number, typically symbolized by Re, is an extremely useful device in fluid mechanics that allows engineers to abstract out a common mathematical problem from physical situations that appear to be different. An important feature of both Reand C_D is that they are dimensionless quantities. For our purposes the Reynolds number is a scalar multiple of the ball speed given by Re = $\frac{\rho L}{\mu} v$ where

$$\rho = 1.23\, \mathrm{kg} \big/ m^3, \text{ mass density of air}$$
$$L = .214 \text{ m, diameter of the ball}$$
$$\mu = 1.79 \times 10^{-5}\, N\, s \big/ m^2, \text{ dynamic viscosity of air}$$
$$v\, m/s = \text{ball speed}$$

This yields the relationship Re = 14705v where v is the ball speed in m/s. Most if not all C_D versus Reynolds number curves look like this one, as can be seen in Fig. 1. The phenomenon at about Re = 100,000 where C_D drops an order of magnitude is call the *drag crisis*. It occurs when the boundary layer of air about 1 mm thick and next to the ball switches from laminar to turbulent. This happens at different Re for different balls as shown in Fig.1. The region of rapidly falling C_D is called the *critical region*. The four data points in the volleyball critical region are shown in Table 1 where v is in m/s.

Table 1

v	C_D
10.2	0.47
15.7	0.15
17.0	0.10
19.7	0.08

The signifigance of the C_D values for a volleyball is that much of the game is played in the critical region. A spike at 30 m/s (Re = 400, 000) is a very hard hit. Serves at 25 m/s are hard and at 20 m/s are common. Along a path of significant length a ball typically slows by about 4 m/s. The result, well known to volleyball players, is that the ball can be made to behave eratically, much like a knuckleball pitch in baseball. This occurs mostly in serves as a result of servers learning to serve a non-spinning ball at crucial speeds vis-a-vis the critical region. It's also important to note that the Hahn-McCulloch data include all speeds that actually occur in volleyball. The data point [1,000,000, .12] corresponds to a speed of 68 m/s and it is likely that no volleyballs are ever hit faster than 34 m/s which is Re = 500,000.

Getting specific with terminal velocity

Some attempts have been made to use this kind of mathematical analysis to provide volleyball coaching insight. Apparently, there is some folklore that all sports balls exhibit a C_D = .2. The only way one could draw this conclusion is to presume that in practice all balls live their lives in the post-critical drag regime at which C_D = .2 and neither assumption is correct for a volleyball. If we were going to simplify C_D to be constant, we would either have to use the value of .47 at the precritical plateau or the postcritical value of .08. We could see what would happen in those two cases. In order to check answers, we will use two methods for computing the terminal velocity:

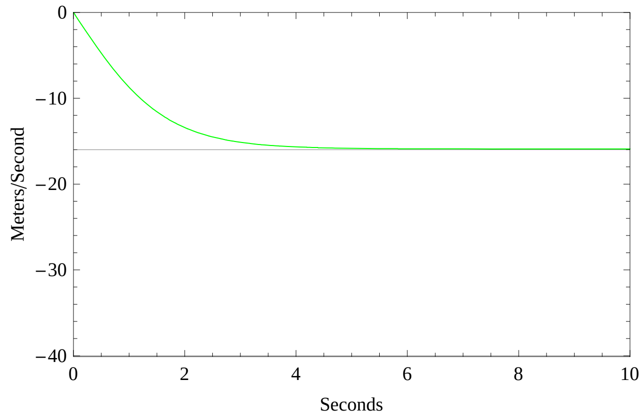
Method 1: Set drag equal to weight and solve for the speed
Method 2: Solve the ODE numerically

Assume C_D = .47

Recall that drag = .5 $\rho A C_D v^2$ and the ball weight is .268×9.8 N = 2.6264 N. Solving drag = weight using *Mathematica*'s FindRoot[ ] produces a terminal velocity of 15.8941 m/s. This could have been solved by hand.

Numerical solution of the differential equation of motion in *Mathematica* using ρ = .47 requires the function NDSolve[ ]. This function produces an interpolating function that can be evaluated at time values within the time range requested of NDSolve[ ] and the result is -15.8939 m/s. Plotting the interpolating function solution produces the graph in Fig. 3.

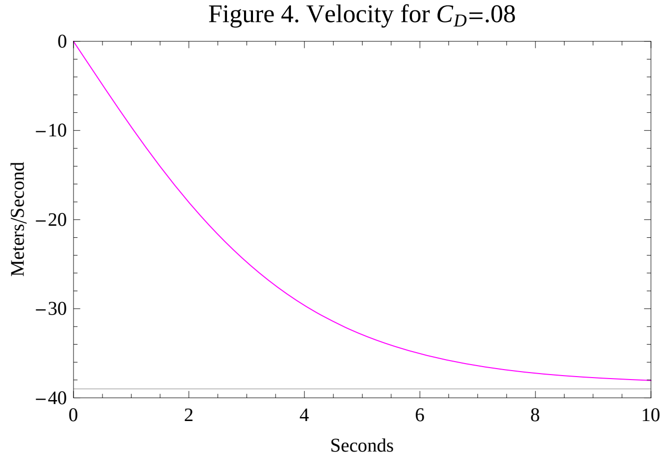
Figure 3. Velocity for C_D=.47



Assume C_D = .08

Solving drag = weight using Mathematica produces a terminal speed of 38.5247 m/s. Solving the differential equation gives us -38.052.

Figure 4. Velocity for C_D=.08

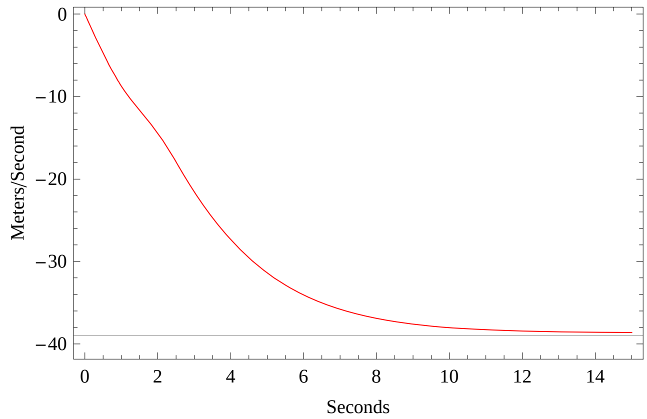


Use C_D values from the Hahn – McCulloch wind tunnel data

Solving weight = drag now requires a numerical algorithm. If we use *Mathematica*'s FindRoot[ ], we must have a function that interpolates the drag instead of the drag coefficient. This is straightforward to produce in *Mathematica* and is included in the Appendix as dragInterp[ ]. The solution found this way is v = 38.7366. Clearly, this approach is inadequate and we should solve the differential equation for a complete set of values.

In this case we ask NDSolve[ ] to call the C_D interpolation function at each iteration. The solution -38.6223 m/s and the graph is shown in Fig. 4.

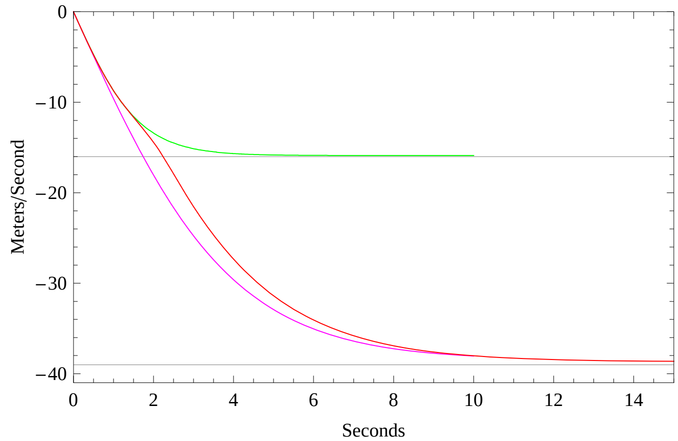
Figure 5. Velocity for Wind Tunnel C_D



Plot all three cases

We can now plot the three cases on the same graph.

Figure 6. Velocities for Three Cases



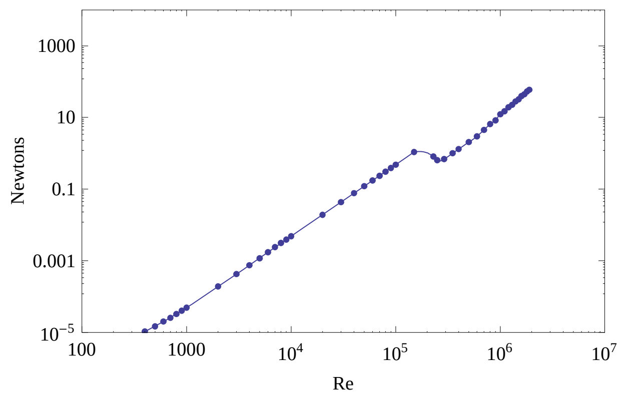
Conclusions

The above computations and plots illustrate what we might have reasoned. If one assumes C_D = .47, the precritical value, the terminal velocity would be almost 16 m/s. But the wind tunnel data shows that the ball encounters the drag crisis prior to its velocity reaching that value. As long as the boundary layer is laminar, the velocity looks almost identical to the C_D = .47 curve. The velocity easily makes it to 20 m/s at which value it has become postcritical and is behaving as if it had the constant value of C_D = .08.

Afterthoughts about drag

Even though C_D drops rapidly at the drag crisis, it's not obvious that the drag also drops. This is because the drag is a multiplicative function of the square of the speed. We might guess that the speed would be increasing rapidly enough that the drag wouldn't decrease even in the face of the drag crisis. To answer this, we can compute the drag from the wind tunnel data and it is plotted in Fig. 7. This shows that the drag actually decreases during the drag crisis.

Figure 7. Drag on a Volleyball



Bibliography

1. Long, Lyle N. and Weiss, Howard, *The Velocity Dependence of Aerodynamic Drag: A primer for Mathematicians*, The American Mathematical Monthly, Vol. 106, No. 2, (Feb., 1999), pp. 127-135.
2. Mehta, Rabindra D. and Pallis, Jani M., *Sports Ball Aerodynamics: Effects of Velocity, Spin and Surface Roughness*. Keynote paper presented at the Materials and Science in Sports Conference, Coronado, California, April 22, 2001.
3. Armenti, Jr., Angelo (editor), *The Physics of Sports*, Springer-Verlag New York, Inc. (1992).
4. Munson, B., Young, D., & Okiishi, T., *Fundamentals of Fluid Mechanics*, John Wiley and Sons, Inc., (1990).
5. McCulloch, David, Beatrice Hahn, *Aerodynamic Forces on a Volleyball.* Unpublished report at U. of Michigan, (1999).
6. Cairns, T., Farrior, D., Farley, A., Marshall, M., Perez, G., Van Lierop, K. and Sigley, A. *Modeling the Path of a Volleyball.* International Journal of Volleyball Research, (2000) 3:1, 2-7.7.
7. Cairns, T., *Modeling Spin and Drag Forces on a Volleyball*, The Engineering of Sport 5. International Sports Engineering Association, (2004).

Biographical Information

Thomas Cairns is a Professor of Mathematical Sciences at The University of Tulsa where he has been faculty since 1959. For seventeen years in the middle of that time span he was varsity volleyball coach and is published in the aerodynamics of the flight of the ball. He can be contacted at cairns@utulsa.edu.