122nd ASEE Annual Conference & Exposition
June 14 - 17, 2015
Seattle, WA
Making Value for Society
Paper ID #14225

Warping Deformation Caused by Twisting Non-circular Shafts

Prof. Somnath Chattopadhyay, University at Buffalo, SUNY

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ABSTRACT

This project is a combined analytical and experimental activity to study warping deformation in shafts of non-circular cross section subjected to torsion. This is a supplemental activity for the junior level Mechanics of Materials course. The students see the warping in square sections and their pronounced effects in ductile materials such as aluminum when the stresses exceed the material yield. The students are made aware of the fact that warping plays a significant role in structural design.

As a part of the project, the students are exposed to the basic theory of torsion as presented in the Mechanics of Materials course. For non-circular cross sections one needs to address warping which is assumed to vary with the rate of twist and is a function only of the position on the cross section and not on the lengthwise coordinate. The students are then exposed to the different ways the warping behavior is modeled analytically through partial differential equations. The warping function model leads to Laplace's equation which is hard to solve even for a rectangular boundaries. Next they are introduced to the St. Venant stress function formulation which leads to the Poisson's equation. This equation has an analytical solution for rectangular cross section in terms of an infinite series involving products of hyperbolic and sinusoidal functions. The expressions for the stress function and its derivatives (the in-plane shear stress components) are then numerically evaluated and plotted for the cross section. The plots show that the shear stresses vanish at the corners, a result that is counterintuitive from the study of torsion of a member of circular cross section. It is inferred that these constraining effects cause the warping deformations to appear in torsion of shafts of non-circular cross-section. A further extension of the work in the experimental area demonstrates pronounced warping as the shear stresses due to the torsional loads exceed the torsional yield strength of the material.

INTRODUCTION

One of the topics in Mechanics of Materials is that of torsion of prismatic sections. The analysis of prismatic bars subjected to axial loads is available for arbitrary cross-sections. In contrast, simple analytical solutions for the deformation and stresses in a bar subject to axial torsion exist only for circular cross-sections. However, it is sometimes necessary to design shafts of non-circular cross-sections. As an example, the design of a connecting rod would require determination of torsional stresses in non-circular sections (typically I-sections). Such situations also exist in the design of various machine parts such as brackets and supports, that are sometimes loaded in torsion. The determination of the stresses and deflections for the torsion of non-circular shafts involve equations that are quite complicated. The assumptions that are valid for circular cross-sections do not apply here. For non-circular shafts under torsion, the plane cross-sections perpendicular to the shaft axis do not remain plane after twisting, and deformation takes place in the axial direction which is called warping. It is not the purpose of this paper to go through the analytical details associated with warping of non-circular shafts under torsion, but instead to develop an appreciation for this important effect through simple experiments.

This activity constitutes an extension of torsion test, typically performed for a torsion specimen of circular cross-section, by adding a test for torsion of a square cross-section. One of the features of the torque-twist relationship was the inherent nonlinearity especially for the specimens with square cross section which was quite pronounced for aluminum. It was difficult to separate the linear part of the torque-twist curve from the experimental results.

WARPING

Warping in square cross-sections was clearly demonstrated by using rubber specimens of square cross-section in which square grids were drawn along the faces of the specimen. Upon twisting the specimens take the shape as shown in Figure 1.

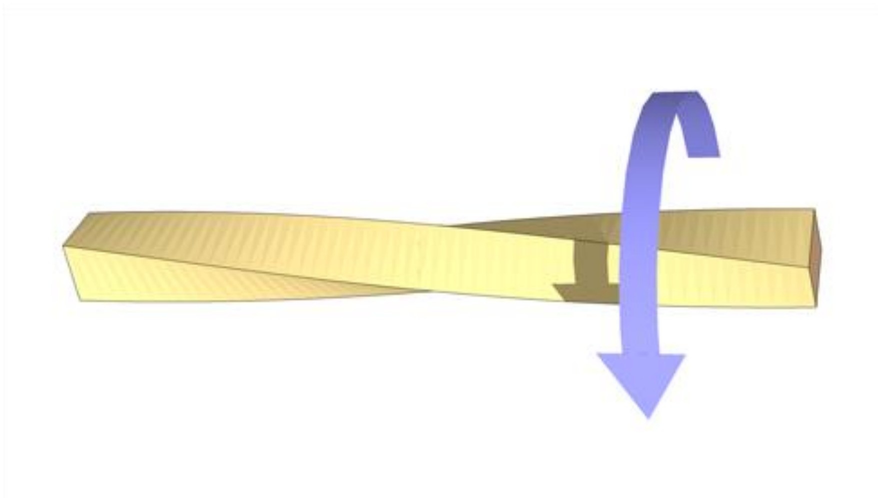

Figure 1 Demonstration of warping in rubber specimens in torsion

Warping was further demonstrated by using the membrane analogy. A steel plate with a square hole was used. Rubber sheet was rigidly clamped at the edges of the hole and made to bulge by applying pressure from beneath the plate. The resulting bulges (torsional hills) for the square hole is shown in Figures 2.

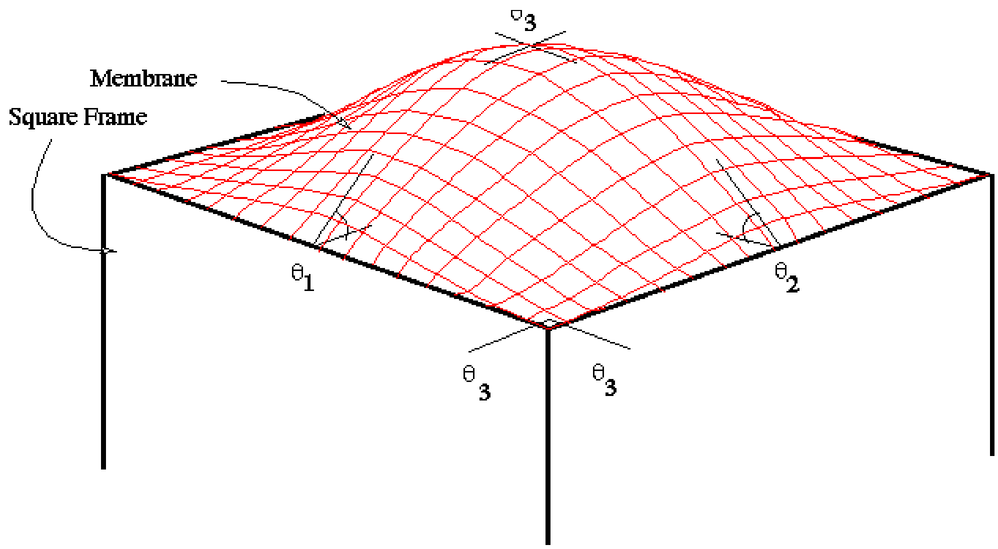

Figure 2 Torsion Hill for Square Cross Section

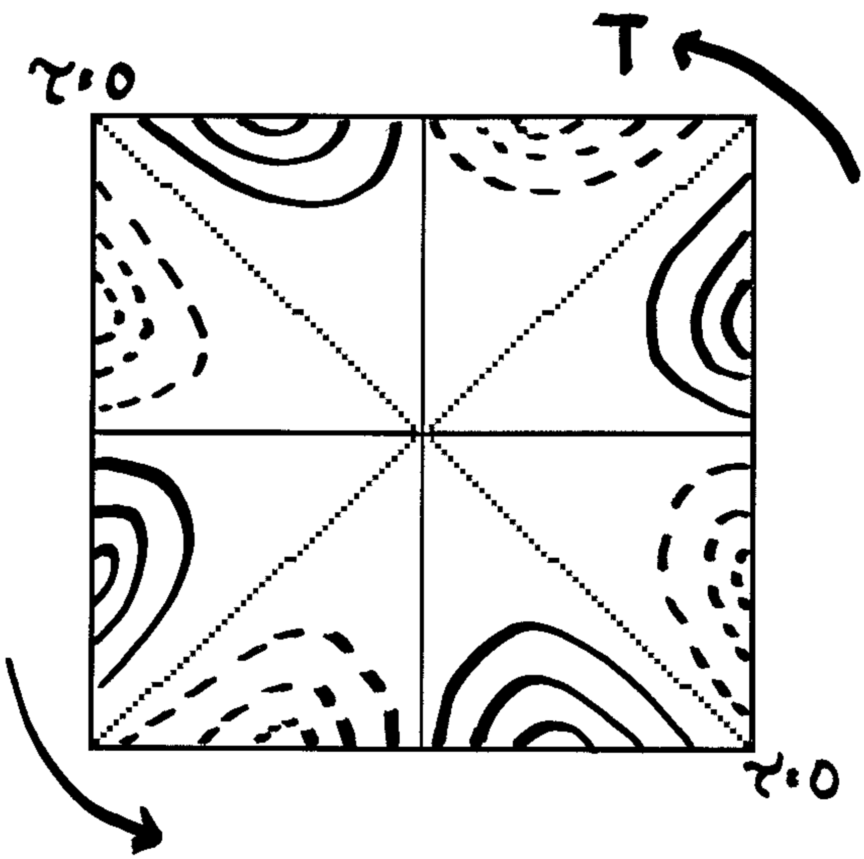

Figure 3 Shear Stress Distributions on a Square Shaft in Torsion

For the square hole, the membrane will be forced to lie flat (zero slope) at the corners and steepest slopes at the midpoints of the outer edges.

The students are alerted to the fact that the analysis of a non-circular torsion member is far more complicated than a circular shaft. It will be wrong to assume that the shearing stress in the cross section of a square bar varies linearly with the distance from the axis of the bar, and thus is largest at the corners of the cross-section. The shear stress is actually zero at the corners. The distorted shape indicative of warping as shown in Figure 1 is due in part to the fact that shear stresses at the corners vanish. The shear stress distribution on a square shaft under torsion is shown in Figure 3.

The students are made aware of the fact that the torsional shear stresses in a shaft of arbitrary cross section are proportional to the slopes of a suitably inflated flexible membrane (membrane analogy). Thus the stress is a maximum at the midpoint of the outer edges of a square shaft, and not at the corners. In the corners the membrane has a zero slope indicating a zero stress.

TORSION EXPERIMENTS

(a) Circular Cross Section

Tinius Olsen torsion tester along with a reaction torque sensor (Figure 4) was used. The torsion tests were performed on 6061extruded aluminum and low alloy steel 1018 bars of circular cross section (0.5 inch in diameter). The loading was increased and continued through the inelastic region. Figure 5 displays the data graphically and shows the torques asymptotically reach constant values for both materials with increase in twist angles.

Figure 4: Experimental Setup

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The maximum shear stress for a solid circular rod due to an applied torque T, is given by [1]

$$\tau_{max} = \frac{Tc}{J} \qquad (1)$$

Where *c* is the distance from the center to the outermost fiber (the outside radius of the shaft), and *J* is the polar moment of inertia. The angle of twist, θ is given by [1]

$$\theta = \frac{T\,L}{G\,J} \qquad (2)$$

Where L is the shaft length, and G is the shear modulus of the shaft material.

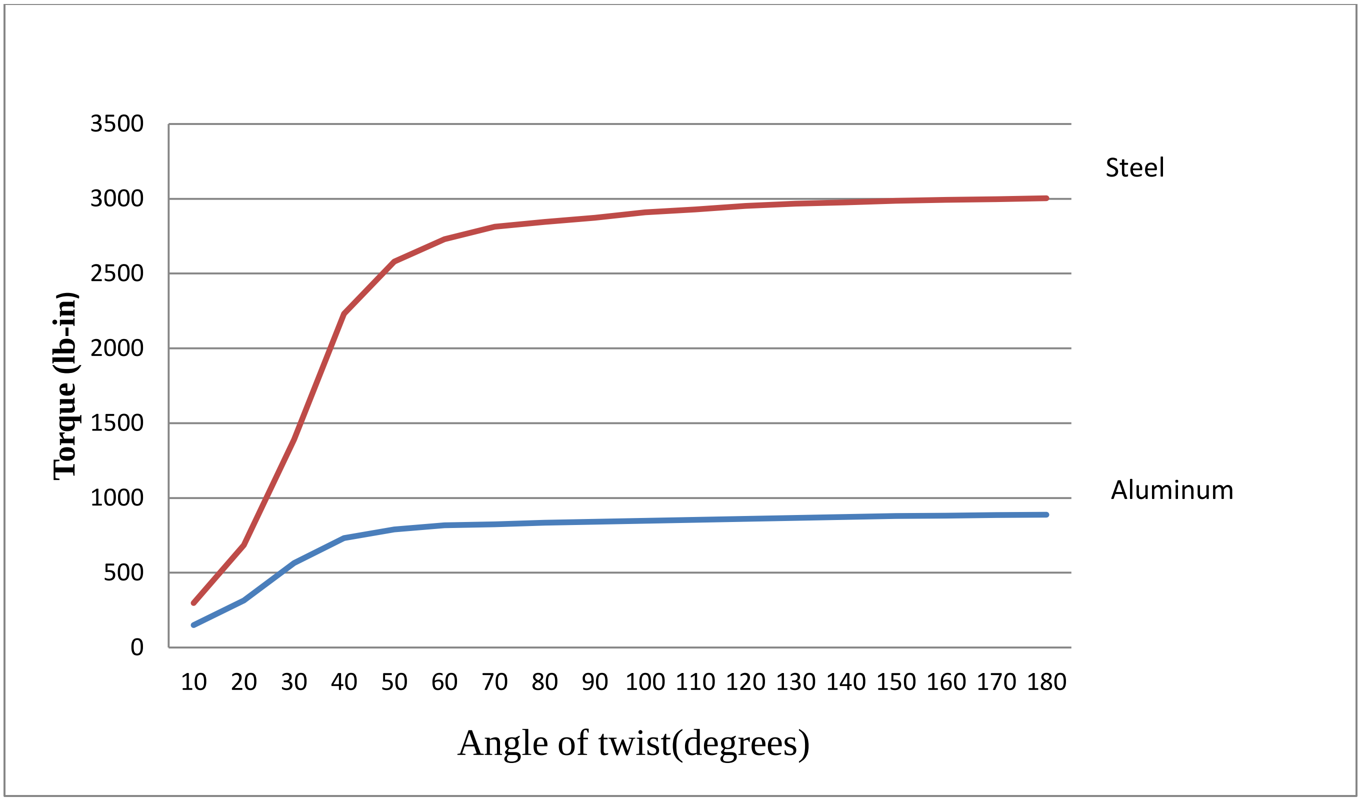

Figure 5: Torque vs. Twist angle for ½ in. diameter aluminum and steel rods.

It is seen from Figure 5 that for the same torque, the aluminum specimen twists considerably more that the steel specimen of the same length (L) and cross section (J) because of reduced shear modulus G of aluminum as compared to steel. The linear part of the torque-twist curves for both materials extend up to a twist angle of 20 degrees or less. This part can be used to verify equation (2), and obtain the elastic stiffness, *GJ/L*.

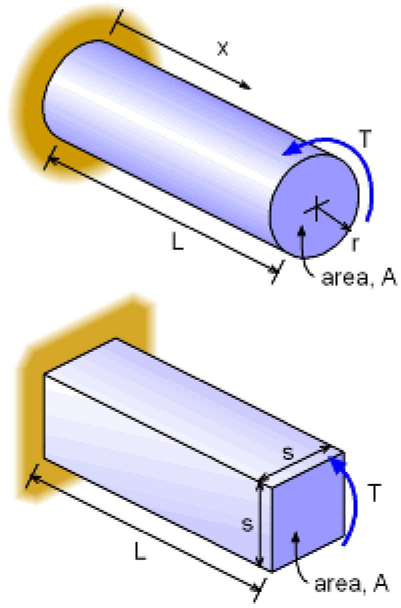

Figure 6: Circular and Square Cross Sections

Figure 6 shows a comparison of the two cross sections analyzed in this work. This section addressed the circular cross section. In the next section, the square cross section is addressed. Note that for this work, a = s = ½ in.

(b) <u>Non-Circular Sections</u>

For this purpose, specimens of a square cross section (½ in x ½ in) of aluminum 6016-T6 were used. The torque-twist characteristics were obtained in the same way as the circular cross section, but only one material was used. The results are shown graphically in Figure 6.

For the square cross section, the determination of torsional stiffness requires consideration of warping which is available only in advanced texts on Mechanics of Materials, such as, Reference [2]. The analytical solution comes of course, from the classical text about stress and strain by Timoshenko and Goodier [3]. For non-circular cross sections one needs to address warping which is assumed to vary with the rate of twist and is a function only of the position on the cross section and not on the lengthwise coordinate. The students are then exposed to the different ways the warping behavior is modeled analytically through partial differential equations. The axial distance (along *x* axis), *w* is only a function of the in plane coordinates (*y*, *z*) and leads to the following equation:

$$\nabla^2 w = 0 \qquad (3)$$

This is Laplace's equation which is hard to solve even for a rectangular boundaries. Next they are introduced to the St. Venant stress function Φ (the in-plane shear stresses are partial derivatives with respect to *y* and *z*) which obeys the Poisson's equation.

$$\nabla^2 \Phi = -2G\theta_L \qquad (4)$$

Where G is the material shear modulus and θ_L is the angle of twist per unit length.

The Poisson equation of equation (4) has an analytical solution for rectangular cross section in terms of an infinite series involving products of hyperbolic and sinusoidal functions. The expressions for the stress function and its derivatives (the in-plane shear stress components) are then numerically evaluated and plotted for the cross section. The plots show that the shear stresses vanish at the corners, a result that is counterintuitive from the study of torsion of a member of circular cross section. The maximum shear stresses are at the midpoints of the sides. It is inferred that these constraining effects cause the warping deformations to appear in torsion of shafts of non-circular cross-section.

For a square cross section, the following results are obtained [3] for a square section of dimension *(a x a)* with the applied torque, T, the twist, θ and the maximum shear stress τ_{max}

$$\tau_{max} = T/(k_1 a^3) \qquad (5)$$

$$\theta = TL/(k_2 a^4 \mathrm{G}) \qquad (6)$$

For a square cross section, $k_1 = 0.208$ and $k_2 = 0.1406$

The corresponding values for a circular cross section of diameter *a* are given by,

$$\tau_{max} = 16T/(\pi a^3) \qquad (7)$$

And

$$\theta = 32TL/(\pi a^4 G) \qquad (8)$$

A further extension of the work in the experimental area demonstrates pronounced warping as the shear stresses due to the torsional loads exceed the torsional yield strength of the material.

The experimental results on the torsion of a square shaft also demonstrated warping which was clearly visible when the square shaft was plastically deformed.

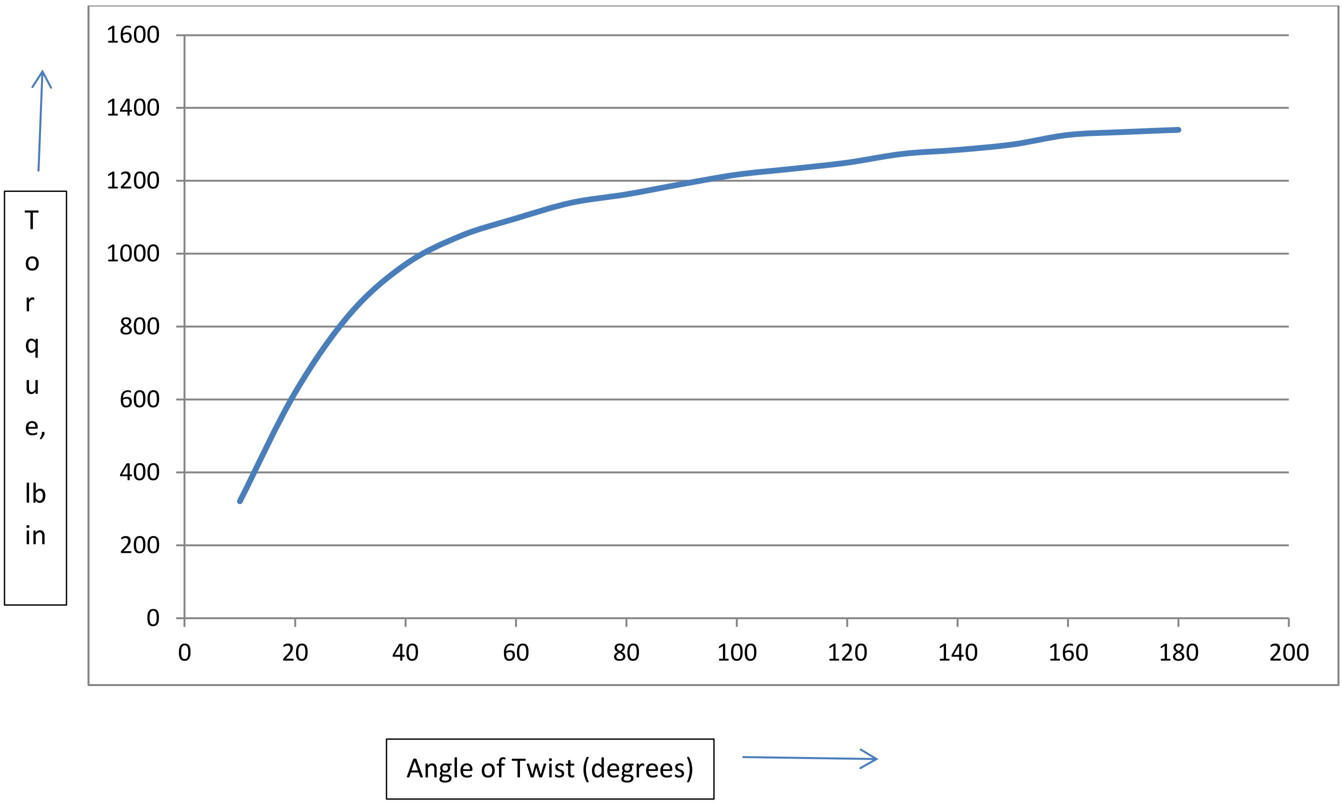

Figure 7. Torque versus Angle of Twist for Square Torsion Specimen

COMPARISON WITH EXPERIMENTAL RESULTS

The results obtained for aluminum specimens of ½ in diameter (circular) and ½in x ½ in (square) have been used for comparing with the theoretical values in equations (5) through (8). These correspond to the lower curve of Figure 5.

The torque twist characteristics corresponding to the linear portion of the curves of Figures 5 and 7 yield the following values for the two cross-sections for the torque per unit twist (in lb-in/degree)

For the circular cross section from equation (6),

$$|T/\theta|_{circle} = \pi a^4 G/(32L) \qquad (7)$$

And that for square cross section from (4),

$$|T/\theta|_{square} = k_2 a^4 G/L \qquad (8)$$

The ratio of the above two torques per unit twist $k_{T\theta}$ is

$$Theoretical \quad k_{T\theta} = \text{square/circle} = 32k_2/\pi = 1.432$$

The experimental value from Figures 5 and 7 is *(32 in-lb/degree)/(15.5 in-lb/degree) = 2.10*

The discrepancy in the values may be attributed to the error of torque twist characteristics at small values of torques due to backlash in the system as well as residual plastic deformation.

As a comparison for the maximum shear stress we equate the stresses obtained in equations (3) and (5), both to the yield conditions corresponding to the asymptotic values of the torque from Figures 5 and 7. For the circular and the square cross sections these values are 888 in-lbs and 1340 in-lbs respectively.

We recognize that the maximum stresses are bounded by the material yield strength. Thus equating the maximum shear stresses (stresses corresponding to the maximum asymptotic values of the torques) for the circular and square cross-sections, the torque ratio is obtained as:

$$T_{square}/T_{circle} = 16\, k_1/\pi = 16(0.208)/\pi = 1.06$$

As a comparison the limiting torque ratio is 1340/888 = 1.51

This comparison of course ignores the shape factor of the square cross section, especially for the predominantly plastic condition at the asymptotic values of the torque. This is evident from the resulting permanent deformation produced for the specimen of square cross section as shown in Figure 7

CONCLUSIONS

As can be seen from Figure 7, a considerable amount of plastic deformation is taking place as the square aluminum rod is twisted. When the specimen is removed from the machine after a torque of 1380 lb-in was applied, it was subject to a very large plastic deformation causing it to deform permanently similar to what is shown for a rubber specimen in Figure 1

Warping has been demonstrated using (a) twisting rubber specimens, (b) membrane analogy, and (c) torsion experiments involving square shafts. Membrane analogy provides a mental picture of the state of stress as shown in Figure 2. The membrane for a square shaft has been shown to have the greatest slope at the midpoint of the edges and not at the corners as is sometimes supposed. In the corner the membrane has a zero slope along both edges indicating zero stress. Experimental investigation on the torsion of square shaft provides a basis of comparison of its torsional stiffness with the analytical values that take warping into consideration. Future efforts in this area would be to measure strains using strain gages to compare with the analytical expressions for shear stresses.

DISCUSSION

This project is an attempt to demonstrate warping in a square shaft under torsion. The students can appreciate how the section warps after the square section specimen is removed from the experimental setup. Because of the accompanying significant plastic deformation in the specimen, the specimen permanently warps similar to what is shown in Figure 1. For aluminum specimen, this is particularly prominent because of the high ductility of the material. In fact it is very hard to separate the plastic deformation from the warping deformation, even for smaller values of applied torques**.** As the torque is increased a plastic region develops around an elastic core. There are errors introduced in the experiment primarily due to setup repeatability.

ASSESSMENT OF STUDENT LEARNING

A number of activities in terms of solving problems, and explaining some concepts will be introduced.

1. Show that if the yield strength is exceeded equations (5) and (7) do not hold. Why? Calculate the maximum elastic torque when the shear yield is given for both square and circular shafts. Also calculate the maximum elastic twist for both shafts.

2. A 25 mm square shaft is made of steel having a shear yield stress of 207 MPa. It is used to transmit power. Determine the maximum torque that can be transmitted. What is the maximum transmitted power, if the shaft runs at 1500 rpm?

3. For the same cross-sectional area and applied torque, what is the ratio of maximum shear stress in the square rod to the circular rod?

4. A new car engineer notices that solid square steel rods are less expensive per pound to purchase than solid circular steel rods. He thinks he can save the company money by specifying square rods for drive shafts in all vehicles. Each rod has the same cross sectional area and same torque is applied to both rods of the same material. What should be the percentage reduction in price needed to make his proposal viable?

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