Wonder, Discovery and Intuition in Elementary Mathematics

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Abstract

A major problem today concerns educating the next generation of engineers, mathematicians and researchers. The value of our concentration on drilling and testing appears questionable. Some students perform well on tests but do not understand why the various algorithms work. Others do poorly, become overwhelmed and give up with feelings of hopelessness. Conceivably, computational ability may not reflect mathematical insight or be a reliable measure of creativity. With readily available calculators, the next generation may not be well served if young people are trained and judged on the speed and accuracy of their computations. Perhaps we should be nurturing analytical judgment and the ability to recognize errors.

Unfortunately, current public discussion involves repeated testing and not the genuine joy of discovery. Areas of mathematics contain wonderful concepts and ideas, which can pique the natural curiosity of young students and enchant them into furthering their mathematical studies. The inner structure of mathematical objects, properly introduced, should suffice to captivate young minds.

My proposition is that with appropriate additional explanation, many youngsters could comprehend and take delight in visual presentations of mathematical concepts. Why not start the analytical education of our children early in elementary school? Accordingly, I have been designing sets of slide shows with the aim of providing a visual framework, which will illuminate the essence of some mathematical concepts.

Each of the slide shows introduces a surprising mathematical fact. In the spirit of mathematical strategy, the progression of slides builds on established facts to explore the unknown. It is hoped that students who view the slides might develop the insight, intuition and confidence needed to successfully explore other analytical situations.

Introduction

"Why do so many people say they hate mathematics?" David Acheson asks in his book 1089 and All That - A Journey into Mathematics[3]. Acheson answers, "All too often, the real truth is that they have never been allowed anywhere near mathematics…" Too many people in their childhood learned to believe that mathematics is mostly arithmetic or algebraic computations, and in the end punishment results from making mistakes or not being capable enough to either remember the procedures or solve the problems. In today's world, calculators and computers can do the drudgery that both non-mathematicians and mathematicians alike would rather avoid. A noble goal would be to teach students to seek understanding of the processes, whether biological, technical, economical, social, mathematical etc. that are evolving around them. It may well be that there is little connection between the ability to memorize the authoritarian rules and procedures of mathematics and the ability to question and wonder, why and how things work.

Effective researchers have a clear idea as to what is known and what is unknown and what remains to be understood. These researchers are comfortable recognizing that they do not know, and are willing to experiment to discover possible new results. Effective researchers understand that science constitutes an ongoing battle against ignorance.

If it is in the nation's interest to promote creative science and research, then our youngsters should be encouraged to explore, to experiment and to make mistakes without punishment. In many disciplines experimentation requires expensive and difficult procedures, but mathematics remains available to everyone to ponder and contemplate at any time and can serve as a model for researching other disciplines.

Forty years ago, mathematicians exhibited pride in their ability to hide geometric or visual representations of mathematical concepts. The mathematics reform of the 1990's sanctioned the representation of functions as single valued curves. More remains to be done. Currently, Proof Without Words is an ongoing feature of the MAA monthly, The College Mathematics Journal. Examine the wonderful compilations, Proofs Without Words [1] and Proofs Without Words II [2] by Roger B. Nelsen. Examine also Math Made Visual by Claudi Alsina & Roger B. Nelson [4]. While mathematicians may enjoy the puzzles provided by Proofs Without Words, an effective pedagogical tool requires verbal explanation. In addition, many of the PWW diagrams are presented as isolated facts. The slide shows enable a teacher to elaborate on the brief comments provided. The slides constitute a progression of ideas, perhaps leading a student to recognize his capacity to understand the development of mathematics.

This slide exhibition consists of four independent slide sets: 1) Fast Counts, 2) Polygons and Circles, 3) Basic laws and 4) Illusion. Each of the slide sets intends to illustrate and develop surprising or curious mathematical phenomena.

1. Fast Counts

Performing a count may not be either simple or easy. Imagine taking a census or managing an election. The slides illustrate how, in special cases, objects can be grouped or arranged in geometric patterns to facilitate counting. If a complex count is needed, a little thought beforehand can simplify the process and minimize the possibility of errors. The sequence illustrates the advantages of grouping, performing computations on related counts, and arranging the objects to be counted in geometric patterns. Objects arranged in rows and columns can be enumerated by performing a multiplication. Objects arranged in a particular triangular pattern can be enumerated by applying the formula $n*(n+1)/2$. Formulas can be derived for objects arranged in combinations of these rectangles and triangles such as trapezoids, hexagons and other polygons.

2. Polygons and Circles

An angle is inscribed in a circle. If the vertex moves on the circle, how does the vertex angle change? Surprisingly, the angle does not change. A beautiful visual proof is illustrated [3] of the case where the sides of the angle extend to the ends of a diameter. Even more surprising is the extension to the case where the sides of the inscribed angle intercept any fixed arc. Again the angle is constant.

Since a quadrilateral can be viewed as two adjacent triangles, the sum of the four angles of any quadrilateral always equals 360^{o}. It is expected that there may be no circle that contains the four vertices of a general quadrilateral. In the special case where the quadrilateral can be inscribed in a circle, it is shown that the sum of the angles at opposite vertices is always 180^{o}.

If the ends of two non-intersecting chords of a circle are connected by two intersecting chords, two triangles are formed. A visual demonstration shows that these triangles are always similar. Similarity of geometric figures means that corresponding angles of the figures are equal as are the ratios of corresponding sides. Based on this property, the slides show that the length of any perpendicular drawn from a point on the diameter to the circle equals the geometric mean of the segments in which the diameter is cut. And based on this fact, the important right triangle law, named after Pythagoras can be visually demonstrated. (If the word "Pythagoras" turns students away, it should be discarded and the Pythagorean theorem renamed after its place in geometric theory, say the slant distance theorem.)

3. Basic laws

Squares of whole numbers are rare. Less than 1% of the whole numbers between 1000 and 9999 are perfect squares. The algorithm $n(n + 1)(n + 2)(n + 3) + 1$ appears to always produce perfect squares. The slides exhibit visually [1] that the algorithm can be derived from the basic laws of numbers and geometry.

The slides establish correspondences between the basic laws of arithmetic and geometric figures. Initially, the Commutative Law of Addition is illustrated by a line diagram. Next, rectangular area diagrams illustrate the Commutative Law of Multiplication. Illustrations are provided that justify The Distributive Law of Multiplication over Addition.

The Binomial Square law, $(a + b)^2 = a^2 + 2ab + b^2$ is presented as a corollary to the Distributive Law. The common misconception of beginning students that $(a + b)^2$ might equal $(a^2 + b^2)$ is definitively disproved by a rectangular area diagram. And finally, the wonderful identity:

$$n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$$

is illustrated by an equally wonderful rectangular area diagram. This last area diagram suggests extensions of the identity.

4. Illusion

Math teachers hid visual representations because, they said, pictures could be misleading. This slide set presents an example of a misleading picture. However, if visual representations can entice youngsters to study math, teachers should actively seek these representations and not fear to employ them as pedagogical tools, maybe with a warning, in those rare cases where a warning might be warranted.

A rectangle of dimensions 5x13 is cut up into two triangles and two trapezoids. The pieces are rearranged to form an 8x8 square. This cannot be; $5 \times 13 = 65$ which does not equal $8 \times 8 = 64$. Areas must be preserved when any polygon is cut up and the pieces rearranged. What is wrong? Maybe the lengths are not right. In addition maybe the angles are not right. Examine the angles. In order for the figure to be true, the diagonal of the rectangle must form the same angle with the horizontal as the angle of the triangle cut from the square and as the slant side of the trapezoid cut from the square. The following geometrical fact is displayed. Two right triangles have equal acute angles if and only if the ratios of corresponding sides are equal. The ratios of corresponding sides are computed and found to be not equal. Therefore, the pieces of the rectangle are not congruent to the pieces of the square. A slide is included which shows how the pieces would look if the deviations were magnified. The conclusion is that the pieces do not fit. The angles are very close, providing an illusion that only appears to fit.

Conclusion

In our quotidian lives, we are constantly encountering puzzles and surprises. It should be expected that puzzles and surprises could be found in elementary mathematics. An opportunity will be lost if these amusing puzzles and surprises are not used to capture the interest of young students in analytical studies. But there is more to mathematics than isolated puzzles and surprises. The puzzles of mathematics lead to other puzzles that lead on to still others. The entire structure of analytical mathematical facts provides the models that prove invaluable in exploring and studying other quantitative fields of knowledge. Mathematics offers the strategies and enables the ability to solve quantitative problems by manipulating symbols.

Einstein is reputed to have said, "Imagination is more important than knowledge. Knowledge is limited. Imagination encircles the world." Today, a great deal of knowledge is easily available. Any eighth grader today can acquire more facts and the contexts surrounding these facts with a few mouse clicks on the Internet than a high school senior just a few decades ago spending a month in a library. Many of the disconnected facts and formula of elementary mathematics derive from principles that can be visually represented. Mathematicians must set aside their distrust of visual models. Our educational system requires employing these representations to instill wonder and insight and judgment.

Our young students are encouraged, by the pressure to earn respectable grades, not to investigate, develop or even master some elegant powerful theory, but to concentrate on mnemonics and disconnected special purpose, computational tricks. Of course, administrators desiring to improve their institutional test scores will reward those teachers who are more adept at drilling the students. Such a process creates obedient conformists and not independent thinking, questioning researchers. It is not reasonable to expect that after more than a decade of memorization, these students will suddenly become enquiring, understanding and creative inventors and researchers. However, perhaps, students who have some glimpse of an interesting, overreaching theory or development might demand help and receive encouragement in their personal study. Furthermore, if our nation wants more creative engineers, likely candidates must be spotted early and provided with a different kind of schooling, a schooling that will nurture curiosity, wonder and imagination as well as offer a safety net for the mistakes that naturally result from curiosity.

References

1. R. B. Nelson, Proofs without Words: Exercises in Visual Thinking, MAA Washington, 1993
2. R. B. Nelson, Proofs without Words II: More Exercises in Visual Thinking, MAA Washington, 2000
3. David Acheson, 1089 and All That - A Journey into Mathematics, 2003
4. Claudi Alsina & Roger B. Nelson, Math Made Visual, MAA, Washington, 2006

Biographical Information

Throughout his career, Dr. Grossfield has combined an interest in engineering design and mathematics. He earned a BSEE at the City College of New York. During the early sixties, he obtained an M.S. degree in mathematics at night while designing circuitry full time for aerospace/avionics companies. He is a member of ASEE, IEEE, and MAA. The slides can be obtained by contacting him at: ai207@bfn.org