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Work in Progress: Connections Between First-Order and Second-Order Dynamic Systems – Lessons in Limit Behavior

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Abstract

There exists a unique relationship between the natural frequency and damping ratio of a lumped-parameter second-order dynamic system and the time constants of equivalent first-order systems. These first-order systems result in the limit of vanishing stiffness or inertia, with the system then capable of storing only a single type of energy.

To emphasize the correspondence of first-order-like behavior with storage of primarily one type of system energy, a pair of two degree-of-freedom systems, one inertia-dominated and one stiffness-dominated, are presented. Although the governing ordinary differential equations are second-order, these systems are overdamped only. In studying limiting behaviors, the paper raises the question of what it means for a system that can never be underdamped to possess a natural frequency.

The paper shows that expressions for natural frequency and damping ratio can be explicitly written in terms of pairs of time constants that arise naturally from the limiting process. The analysis is presented in a way that is amenable to undergraduate engineering students in courses in system dynamics.

Introduction

In undergraduate system dynamics courses, students are introduced to lumped-parameter, idealized models of mechanical, electrical, fluid, and thermal systems that are governed by analogous ordinary differential equations. For simple models, these equations are either first-order or second-order in time. For example, in a lumped-parameter model of a mass and viscous damper in series the position of the mass is governed by a first-order, linear, ordinary differential equation. From this first-order equation, the time-constant is defined in terms of the mass and damping. In a lumped-parameter mass-spring-damper model of a mechanically translating system the position of the mass is governed by a second-order, linear, ordinary differential equation. From the second-order equation, the natural frequency and damping ratio are defined in terms of the mass, stiffness, and damping.

In the standard presentation, there is limited, if any, connection made between the properties of first-order systems (the time-constant) and second-order systems (the natural frequency and damping ratio). First-order and second-order systems are generally taught as separate cases. This paper shows that first-order systems are limiting cases of second-order systems. The analysis underlying the connections between the properties of these systems is amenable to undergraduate engineering students. The idea provides an alternate and intriguing way to approach core concepts of system dynamics.

Underdamped systems exhibit oscillatory behavior, and expressions for their natural frequency as typically presented have unambiguous meaning. However, as systems become overdamped, their behavior is non-oscillatory and can be characterized by time constants of first-order-like systems. This becomes the case in the limit as second-order systems become dominated by their stiffness or inertia. These limits, derived below, represent overdamped behavior for which the meaning of natural frequency is questionable.

The meaning of natural frequency for second-order systems that do not oscillate, e.g., overdamped systems, is not typically addressed in engineering textbooks [1-5]. It is unclear whether there is agreement in the physics and mechanics communities on a precise and unambiguous definition of a system's natural frequency. In the physics community, the topic has been raised, specifically in terms of what an unambiguous definition of "natural frequency" would offer in clarifying how to present overdamped system behavior to undergraduates [6].

Some basic concept questions can be posed. Do all dynamic systems governed by second-order ordinary differential equations exhibit a "natural frequency" or does the term apply only for undamped and underdamped systems that experience oscillatory behavior? Does it make sense that a mathematical definition of natural frequency exists without knowing if the system will oscillate? Can expressions for undamped natural frequency include dissipative terms? This paper presents a series of example problems in which these questions arise.

Models and Behavior

Mass-Spring-Damper Second-Order System

Consider the classical lumped-parameter linear model of a mass-spring-damper system, depicted in Figure 1, with mass *m*, viscous damping *b*, linear stiffness *k*, and applied force *F*(*t*).

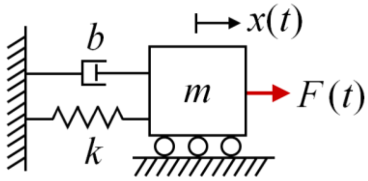

Figure 1. A second-order mass-spring-damper system.

From Newton's second law, the equation of motion in terms of displacement *x*(*t*) is

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)\,, \tag{1}$$

which can be written in normalized form,

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = \frac{1}{m}F(t)\,, \tag{2}$$

in terms of the undamped natural frequency,

$$\omega_n = \sqrt{\frac{k}{m}}\,, \tag{3}$$

and damping ratio,

$$\zeta = \frac{b}{2\sqrt{km}}. \tag{4}$$

The mathematical expression for the natural frequency is written solely in terms of parameters related to energy storage, i.e., kinetic energy (related to *m*) and potential energy (related to *k*).

Focusing on the natural or free (unforced) response, the solution of Eq.(2) in the absence of external excitation or forcing is given by

$$x(t) = C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t), \tag{5}$$

where C_1 and C_2 are constants determined from the initial conditions, λ_1 and λ_2 are the roots of the characteristic equation, and $\exp(\alpha) = e^{\alpha}$ is the natural exponential function.

For an underdamped system for which $0 < \zeta < 1$, the roots are complex conjugates,

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}, \tag{6}$$

where $i = \sqrt{-1}$ and the free response can be expressed as,

$$x(t) = \exp(-\zeta\omega_n t)\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right], \tag{7}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency and *A* and *B* are constants determined from the initial conditions. The underdamped response is oscillatory with an exponentially decaying envelope.

For an overdamped system for which $\zeta > 1$, the roots are distinct and real frequencies,

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}, \tag{8}$$

giving the free response,

$$x(t) = A\exp\left[\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t\right] + B\exp\left[\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t\right] \tag{9}$$

or, in terms of the mass, stiffness, and damping,

$$x(t) = A\exp\left[\left(\frac{-b + \sqrt{b^2 - 4km}}{2m}\right)t\right] + B\exp\left[\left(\frac{-b - \sqrt{b^2 - 4km}}{2m}\right)t\right]. \tag{10}$$

Toward finding a more general definition of natural frequency, the next two subsections of the paper explore the limiting behavior of second-order systems as inertia or stiffness properties dominate.

Inertia-Dominated Second-Order System

As the stiffness of the mass-spring-damper system of Figure 1 becomes smaller, the system becomes inertia-dominated. As the stiffness tends toward zero, the response becomes over-damped and the solution is given by Eqs.(9) or (10). In the limit as $k \to 0$, the exponent in the first term of Eq.(10) tends towards zero (and only a constant term survives) and the exponent in the second term simplifies,

$$\lim_{k\to 0} x(t) = A\exp[0t] + B\exp\left[\left(\frac{-2b}{2m}\right)t\right] = A + B\exp\left[-\frac{t}{m/b}\right] = A + B\exp\left[-\frac{t}{\tau_1}\right], \qquad (11)$$

where $\tau_1 \equiv \dfrac{m}{b}$.

Taking the limit of Eq.(1) with acceleration normalized,

$$\lim_{k\to 0}\left[\ddot{x}(t) + \frac{b}{m}\dot{x}(t) + \frac{k}{m}x(t)\right] = \ddot{x}(t) + \frac{b}{m}\dot{x}(t) = \dot{v}(t) + \frac{1}{\tau_1}v(t), \qquad (12)$$

where $v(t) = \dot{x}(t)$ is the velocity, or taking the limit of Eq.(2),

$$\lim_{k\to 0}\left[\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t)\right] = \ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) = \dot{v}(t) + 2\zeta\omega_n v(t). \qquad (13)$$

Although by definition

$$\frac{1}{2\zeta\omega_n} = \frac{m}{b}, \qquad (14)$$

it is only in the limit of vanishing stiffness that

$$\lim_{k\to 0}\frac{1}{2\zeta\omega_n} = \lim_{k\to 0}\frac{m}{b} = \tau_1. \qquad (15)$$

In summary, the second-order mass-spring-damper system with vanishing stiffness becomes a first-order system with time constant τ_1. In the limit as the stiffness tends toward zero, the inverse of two times the product of the second-order system parameters, ζ and ω_n, i.e., $1/2\zeta\omega_n$, reduces to the first-order system parameter, τ_1. This connection between parameters of first and second-order systems may not be intuitively obvious to an undergraduate student and it is not generally presented in textbooks.

Stiffness-Dominated Second-Order System

As the mass of the mass-spring-damper system of Figure 1 becomes smaller, the system becomes stiffness-dominated. As the mass tends toward zero, the response becomes overdamped and the solution is given by Eqs.(9) or (10). In the limit as $m \to 0$, the second term of Eq.(10) tends exponentially towards zero and only the first term survives,

$$\lim_{m\to 0} x(t) = \lim_{m\to 0} A\exp\left[\left(\frac{-b+\sqrt{b^2-4km}}{2m}\right)t\right], \tag{16}$$

where the exponent tends toward $\frac{0}{0}$ necessitating application of L'Hôpital's rule,

$$\lim_{m\to 0} x(t) = \lim_{m\to 0} A\exp\left[\left(\frac{-b+\sqrt{b^2-4km}}{2m}\right)t\right] = \lim_{m\to 0} A\exp\left[\left(\frac{\frac{d}{dm}\left\{-b+\sqrt{b^2-4km}\right\}}{\frac{d}{dm}\{2m\}}\right)t\right]$$

$$= \lim_{m\to 0} A\exp\left[\left(\frac{\frac{1}{2}\left\{b^2-4km\right\}^{-1/2}(-4kt)}{2}\right)t\right] = A\exp\left[-\frac{t}{b/k}\right] = A\exp\left[-\frac{t}{\tau_2}\right], \tag{17}$$

where $\tau_2 \equiv \frac{b}{k}$.

Taking the limit of Eq.(1) with position normalized,

$$\lim_{m\to 0}\left[\frac{m}{k}\ddot{x}(t) + \frac{b}{k}\dot{x}(t) + x(t)\right] = \frac{b}{k}\dot{x}(t) + x(t) = \tau_2\dot{x}(t) + x(t), \tag{18}$$

or taking the limit of Eq.(2),

$$\lim_{m\to 0}\left[\frac{1}{\omega_n^2}\ddot{x}(t) + \frac{2\zeta}{\omega_n}\dot{x}(t) + x(t)\right] = \frac{2\zeta}{\omega_n}\dot{x}(t) + x(t) \;. \tag{19}$$

Although by definition

$$\frac{2\zeta}{\omega_n} = \frac{b}{k}, \tag{20}$$

it is only in the limit of vanishing mass that

$$\lim_{m\to 0}\frac{2\zeta}{\omega_n} = \lim_{m\to 0}\frac{b}{k} = \tau_2. \tag{21}$$

In summary, the second-order mass-spring-damper system with vanishing mass becomes a first-order system with time constant τ_2. In the limit as the mass tends toward zero the term $2\zeta/\omega_n$, which is a function of the two second-order system parameters, ζ and ω_n, reduces to the first-order system parameter, τ_2. Again, this connection between parameters of first and second-order systems may not be intuitively obvious to an undergraduate student and it is generally not presented in textbooks.

General Second-Order System Expressed in Terms of Time Constants

The governing differential equation, Eq.(1), can be expressed in terms of first-order time constants defined above. With $\tau_1 = \dfrac{m}{b}$ and $\tau_2 = \dfrac{b}{k}$, Eq.(1) can be written as

$$\tau_1 \tau_2 \ddot{x}(t) + \tau_2 \dot{x}(t) + x(t) = \frac{1}{k} F(t). \tag{22}$$

Comparing to the form of Eq.(2), the natural frequency is

$$\boxed{\omega_n = \frac{1}{\sqrt{\tau_1 \tau_2}}} \tag{23}$$

and the damping ratio is

$$\boxed{\zeta = \frac{1}{2}\sqrt{\frac{\tau_2}{\tau_1}}}. \tag{24}$$

Eqs.(23) and (24) hold for both under- and overdamped second order systems and are intriguing. The products and ratios of time constants are directly related to the two characteristic properties of second-order systems. In particular, the natural frequency is the inverse of the square root of the products, i.e., the inverse of the geometric mean of the time constants. The damping ratio is the square root of the ratio of the time constants divided by two. The authors have not seen these equations in system dynamics textbooks.

For the inertia-dominated case, the limit of vanishing stiffness $k \to 0$ corresponds to $\tau_2 \to \infty$. Rewriting Eq.(22) as

$$\tau_1 \ddot{x}(t) + \dot{x}(t) + \frac{1}{\tau_2} x(t) = \frac{1}{b} F(t), \tag{25}$$

and taking the limit $\tau_2 \to \infty$ gives

$$\tau_1 \ddot{x}(t) + \dot{x}(t) = \frac{1}{b} F(t), \tag{26}$$

which is a first-order differential equation in terms of velocity, *v*,

$$\tau_1 \dot{v}(t) + v(t) = \frac{1}{b} F(t). \tag{27}$$

Analogously, for the stiffness-dominated case, the limit of vanishing inertia $m \to 0$ corresponds to the limit of vanishing τ_1. Taking the limit $\tau_1 \to 0$, Eq.(22) becomes

$$\tau_2 \dot{x}(t) + x(t) = \frac{1}{k} F(t). \tag{28}$$

Eqs.(26) and (28) are the equations of the first-order systems that are the limiting cases of the second-order system described by Eq.(22).

Second-Order Systems with Single Type of Energy Storage

Two example problems are presented next in which natural frequency expressions contain dissipative terms. They follow the form of Eq.(23) except the dissipation terms do not cancel. Both problems involve two degree-of-freedom systems that include two types of system elements: one type stores (kinetic or potential) energy and one type dissipates energy. It is known that single degree-of-freedom systems that store only one type of energy are governed by first-order differential equations. These systems cannot oscillate.

Second-Order System Storing Kinetic Energy

A two degree-of-freedom mechanical system that stores only kinetic energy is the series mass-damper system shown in Figure 2.

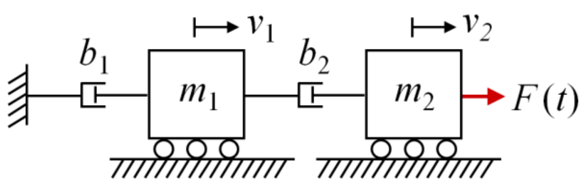

Figure 2. A second-order dual mass-damper system.

This system with masses m_1 and m_2, viscous dampers b_1 and b_2, and applied force $F(t)$ can be viewed as two cascaded first-order systems. The equation of motion for velocity $v_2(t)$ is

$$\ddot{v}_2(t)+\left(\frac{b_1}{m_1}+\frac{b_2}{m_1}+\frac{b_2}{m_2}\right)\dot{v}_2(t)+\left(\frac{b_1}{m_1}\frac{b_2}{m_2}\right)v_2(t)=\left(\frac{b_1+b_2}{m_1m_2}\right)F(t)+\left(\frac{1}{m_2}\right)\dot{F}(t)\,. \tag{29}$$

Comparing to the form of Eq.(2), the “natural frequency” is

$$\omega_n=\sqrt{\frac{b_1b_2}{m_1m_2}}\,. \tag{30}$$

Although the dimensions of this expression are those of frequency, the meaning of natural frequency for this system is not obvious. The system is incapable of oscillating; without springs there can be no transfer of kinetic energy to potential energy. Although the equation of motion is second order and a mathematical expression for “natural frequency” exists, the masses can never oscillate. The existence of a second-order governing differential equation does not mean oscillatory behavior is possible as implied by the word “frequency”. In addition, Eq.(30) contains dissipation parameters, b_1 and b_2. If ω_n represents the undamped natural frequency, then ω_n would be zero.

It is possible to find an expression for the damping ratio in terms of system parameters. Again, comparing Eq.(29) to the form of Eq.(2),

$$2\zeta\omega_n = \frac{b_1}{m_1} + \frac{b_2}{m_1} + \frac{b_2}{m_2}, \tag{31}$$

and solving for the damping ratio ζ,

$$\zeta = \frac{1}{2}\left(\sqrt{\frac{m_2}{m_1}}\sqrt{\frac{b_1}{b_2}} + \sqrt{\frac{m_2}{m_1}}\sqrt{\frac{b_2}{b_1}} + \sqrt{\frac{m_1}{m_2}}\sqrt{\frac{b_2}{b_1}}\right). \tag{32}$$

Introducing the non-dimensional property ratios $\beta = \frac{b_2}{b_1}$ and $\mu = \frac{m_2}{m_1}$, the damping ratio ζ can be expressed as

$$\zeta = \frac{1}{2}\left(\sqrt{\frac{\mu}{\beta}} + \sqrt{\mu\beta} + \sqrt{\frac{\beta}{\mu}}\right) \quad \text{or} \quad \zeta = \frac{1}{2}\left(\frac{\mu + \mu\beta + \beta}{\sqrt{\mu\beta}}\right). \tag{33}$$

From the condition $\frac{\partial\zeta}{\partial\mu} = 0$ for fixed β, the minimum value of ζ occurs at

$$\mu = \frac{\beta}{\beta + 1} \tag{34}$$

and is given by

$$\zeta_{\min} = \sqrt{1+\beta}\ . \tag{35}$$

Since $\beta = \frac{b_2}{b_1} \geq 0$ then $\zeta_{\min} = \sqrt{1+\beta} \geq 1$, proving the response cannot be underdamped. From the condition $\frac{\partial\zeta}{\partial\beta} = 0$ for fixed μ, the minimum value of ζ occurs at

$$\beta = \frac{\mu}{\mu + 1} \tag{36}$$

and is given by

$$\zeta_{\min} = \sqrt{1+\mu}\ . \tag{37}$$

Since $\mu = \frac{m_2}{m_1} \geq 0$ then $\zeta_{\min} = \sqrt{1+\mu} \geq 1$, again proving the response cannot be underdamped.

A three-dimensional plot of damping ratio vs. mass ratio and damping coefficient ratio is given in Figure 3. It confirms that ζ values less than one do not exist. The symmetry of the plot reflects the interchangeability of β and μ in Eq.(33).

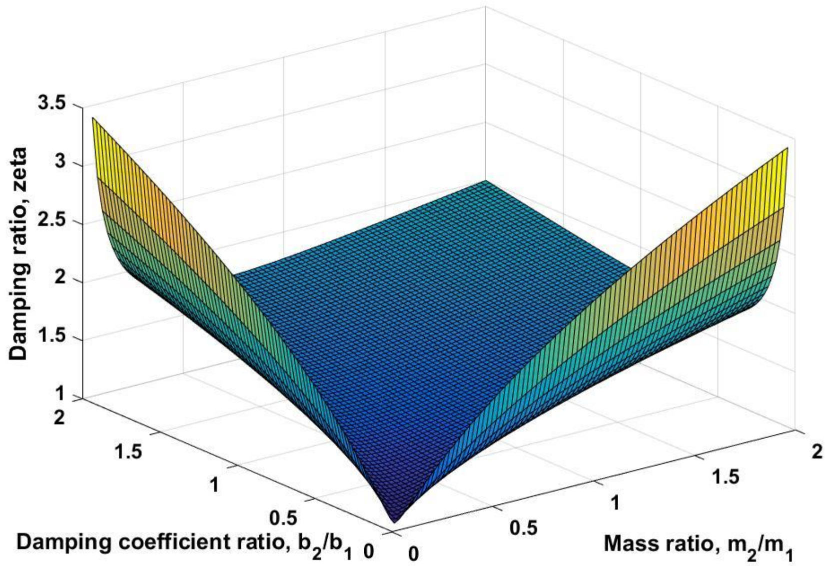

Figure 3. Damping ratio vs. property ratios (mass ratio and damping coefficient ratio) in a second-order dual mass-damper system.

Introducing time constants $\tau_1 = \frac{m_1}{b_1}$ and $\tau_2 = \frac{m_2}{b_2}$, Eq.(30) for the natural frequency can be written as

$$\omega_n = \frac{1}{\sqrt{\tau_1 \tau_2}}, \tag{38}$$

which matches Eq.(23), and from Eq.(29),

$$2\zeta\omega_n = \tau_1 + (1+\beta)\tau_2 \quad \text{or} \quad 2\zeta\omega_n = (1+\mu)\tau_1 + \tau_2 \tag{39}$$

giving expressions for the damping ratio,

$$\zeta = \frac{\tau_1 + (1+\beta)\tau_2}{2\sqrt{\tau_1\tau_2}} \quad \text{or} \quad \zeta = \frac{(1+\mu)\tau_1 + \tau_2}{2\sqrt{\tau_1\tau_2}}, \tag{40}$$

which may be recast in time constant ratio form as

$$\zeta = \frac{1}{2}\left(\sqrt{\frac{\tau_1}{\tau_2}} + (1+\beta)\sqrt{\frac{\tau_2}{\tau_1}}\right) \quad \text{or} \quad \zeta = \frac{1}{2}\left((1+\mu)\sqrt{\frac{\tau_1}{\tau_2}} + \sqrt{\frac{\tau_2}{\tau_1}}\right). \tag{41}$$

Again, ζ can be written explicitly as a ratio of time constants. Compared to Eq.(24), the damping ratio expressions of Eq.(41) are parameter ratio weighted sums of (square roots of) time constant ratios. The minimum values for ζ are given by Eqs.(35) and (37).

Second-Order System Storing Electrostatic Potential Energy

A second-order system that stores only electrostatic energy is the dual series *RC* electrical circuit, shown in Figure 5, with resistances R_1 and R_2, capacitances C_1 and C_2, and voltage source, $e_s(t)$.

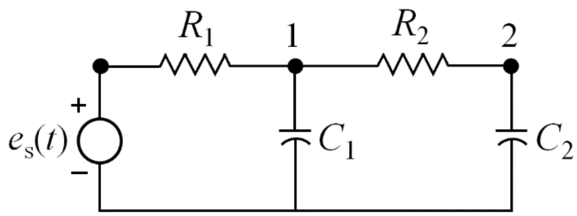

Figure 5. A second-order dual RC circuit

The differential equation for the voltage across capacitor C_2, $e_2(t)$, is

$$\ddot{e}_2(t)+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}\right)\dot{e}_2(t)+\left(\frac{1}{R_1C_1R_2C_2}\right)e_2(t)=\left(\frac{1}{R_1C_1R_2C_2}\right)e_s(t)\,. \tag{42}$$

Comparing to the form of Eq.(2), the natural frequency is

$$\omega_n=\sqrt{\frac{1}{R_1C_1R_2C_2}}\,. \tag{43}$$

Again, an expression for "natural frequency" exists for a system that is incapable of an oscillatory response. It contains dissipative parameters, R_1 and R_2. In this case, if the resistances vanish, Eq.(43) predicts infinite natural frequency.

It is also possible to find an expression for the damping ratio,

$$\zeta=\frac{1}{2}\left(\sqrt{\frac{R_2C_2}{R_1C_1}}+\sqrt{\frac{R_1C_2}{R_2C_1}}+\sqrt{\frac{R_1C_1}{R_2C_2}}\right). \tag{44}$$

Introducing the non-dimensional property ratios $\rho=\dfrac{R_2}{R_1}$ and $\sigma=\dfrac{C_2}{C_1}$, Eq.(44) can be written as

$$\zeta=\frac{1}{2}\left(\sqrt{\rho\sigma}+\sqrt{\frac{\sigma}{\rho}}+\sqrt{\frac{1}{\rho\sigma}}\right)\quad\text{or}\quad\zeta=\frac{1}{2}\left(\frac{1+\sigma+\rho\sigma}{\sqrt{\rho\sigma}}\right). \tag{45}$$

From the condition $\dfrac{\partial\zeta}{\partial\sigma}=0$ for fixed ρ, the minimum value of ζ occurs at

$$\sigma=\frac{1}{1+\rho} \tag{46}$$

and is given by

$$\zeta_{\min} = \sqrt{1+\frac{1}{\rho}} \ . \tag{47}$$

Since $\rho = \frac{R_2}{R_1} \geq 0$ then $\zeta_{\min} = \sqrt{1+\frac{1}{\rho}} \geq 1$, proving the response cannot be underdamped.

A three-dimensional plot of the damping ratio as a function of capacitance ratio and resistance ratio is shown in Figure 4. It confirms that there are no ζ values less than one. (In contrast to Figure 3, the plot is not symmetric; ρ and σ are not interchangeable in Eq.(45).)

In summary, this section of the paper has presented two example problems that are modeled as second-order systems that are always overdamped. In these examples the expressions for natural frequency depend explicitly on dissipation terms. In the following section a general second-order example with two types of energy storage is considered. The expression for natural frequency in this problem depends explicitly on dissipation terms in ratio form.

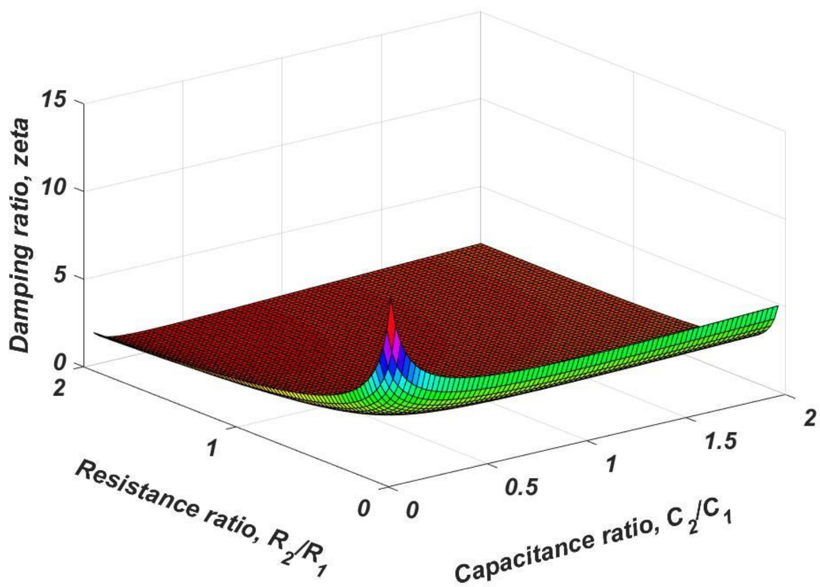

Figure 4. Damping ratio vs. property ratios (resistance ratio and capacitance ratio) in a second-order dual *RC* circuit

Second-Order System with Two Types of Energy Storage

A physically motivated model is a motor that turns a flywheel via a long, elastic shaft supported by a pair of bearings, as shown in Figure 5. The motor shaft rotates at angular velocity $\Omega_M(t)$ and the flywheel at angular velocity $\Omega_F(t)$. The motor is modeled as a torque source, $T(t)$, the long shaft as a torsional spring with stiffness, K, the bearings as viscous torsional dampers with damping B_1 and B_2, and the flywheel as an inertia, J. This system contains two types of energy storage (potential energy in the shaft and kinetic energy in the flywheel) and may exhibit oscillatory behavior.

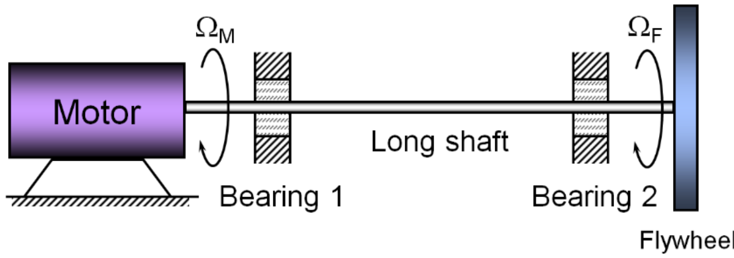

Figure 5. Motor-Shaft-Flywheel System

The equation of motion for the flywheel angular velocity, $\Omega_F(t)$, is[1]:

$$\ddot{\Omega}_F(t) + \left(\frac{B_2}{J} + \frac{K}{B_1}\right)\dot{\Omega}_F(t) + \left(1 + \frac{B_2}{B_1}\right)\frac{K}{J}\,\Omega_F(t) = \left(\frac{K}{JB_1}\right)T(t)\,. \tag{48}$$

Comparing to the form of Eq.(2), the natural frequency is

$$\omega_n = \sqrt{\left(1 + \frac{B_2}{B_1}\right)\frac{K}{J}}\,. \tag{49}$$

It is atypical in most undergraduate system dynamics courses to encounter expressions for natural frequency that depend on dissipative system parameters, even appearing in ratio form. Some textbooks present a definition of natural frequency as the frequency at which unforced systems oscillate in the absence of dissipation. This example shows that the natural frequency is a function of a ratio of damping terms. In the previous examples, the natural frequency expressions were functions of dissipation terms (b_1, b_2 in Eq.(30), R_1, R_2 in Eq.(43)).

It is also possible to find an expression for the damping ratio. Again, comparing to the form of Eq.(2),

$$2\zeta\omega_n = \frac{B_2}{J} + \frac{K}{B_1}, \tag{50}$$

and solving for the damping ratio ζ,

$$\zeta = \frac{1}{2}\frac{1}{\sqrt{\eta}}\left(\frac{B_2}{\sqrt{JK}} + \frac{\sqrt{JK}}{B_1}\right) \tag{51}$$

where $\eta = 1 + \dfrac{B_2}{B_1}$.

Introducing the time constants $\tau_1 = \dfrac{B_1}{K}$ and $\tau_2 = \dfrac{J}{B_2}$, the differential equation can be written,

[1] This differential equation is identical to one governing blood flow rate in a 4-element Windkessel model for arterial blood flow [7].

$$\ddot{\Omega}_F(t) + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\dot{\Omega}_F(t) + \left(1 + \frac{B_1}{B_2}\right)\left(\frac{1}{\tau_1\tau_2}\right)\Omega_F(t) = \left(\frac{1}{B_2}\right)\left(\frac{1}{\tau_1\tau_2}\right)T(t)\,, \tag{52}$$

giving the natural frequency

$$\omega_n = \sqrt{\left(1 + \frac{B_1}{B_2}\right)\frac{1}{\tau_1\tau_2}}\,, \tag{53}$$

and

$$2\zeta\omega_n = \frac{1}{\tau_1} + \frac{1}{\tau_2}\,, \tag{54}$$

which can be solved for the damping ratio,

$$\zeta = \frac{1}{2}\sqrt{1 + \frac{B_1}{B_2}}\left(\sqrt{\frac{\tau_1}{\tau_2}} + \sqrt{\frac{\tau_2}{\tau_1}}\right). \tag{55}$$

When expressed in terms of time constants, the limits of stiffness-dominated and inertia-dominated systems are straightforward and give rise to first-order differential equations. Eq.(52) can be written equivalently as,

$$\frac{\tau_1\tau_2}{\left(1 + \frac{B_1}{B_2}\right)}\ddot{\Omega}_F(t) + \frac{\tau_1 + \tau_2}{\left(1 + \frac{B_1}{B_2}\right)}\dot{\Omega}_F(t) + \Omega_F(t) = \left(\frac{1}{B_1 + B_2}\right)T(t)\,. \tag{56}$$

In the stiffness-dominated regime ($\tau_2 \to 0$), from Eq.(56),

$$\frac{\tau_1}{\left(1 + \frac{B_1}{B_2}\right)}\dot{\Omega}_F(t) + \Omega_F(t) = \left(\frac{1}{B_1 + B_2}\right)T(t)\,. \tag{57}$$

In the inertia-dominated regime ($\tau_1 \to \infty$), from Eq.(52),

$$\tau_2\dot{\alpha}_F(t) + \alpha_F(t) = 0\,, \tag{58}$$

where $\alpha_F(t) = \dot{\Omega}_F(t)$.

Discussion

Limit Behavior

The paper has shown that there is a continuous limit of second-order behavior that exhibits first-order behavior when the damping ratio is sufficiently large. In the continuous limit of unbounded damping ratio, there exists a unique relationship between the natural frequency and damping ratio of the second-order system and the time constant of an equivalent first-order system. For mechanical systems, an unbounded damping ratio corresponds to either vanishing stiffness or vanishing inertia, with the resulting system capable of storing only a single type of energy, either kinetic or potential energy.

The analysis is amenable to undergraduate engineering students in courses in system dynamics. One author (VP) has offered the exercise of limiting behavior of inertia-dominated and stiffness-dominated second-order systems (for extra credit) to students for over a decade. The goal is for students to find the first-order-like time constants presented here[2].

We have not seen this presentation in textbooks that cover dynamics, system dynamics, and modeling [1-5], although it appears, in part, in a thorough discussion by Rowell [8] in course notes at MIT.

Meaning of Natural Frequency

Despite the often-accepted understanding that all second-order systems possess a natural frequency, only second-order systems with two independent and different energy storage elements can exhibit responses with a natural frequency. An example is the mass-spring-damper system for which kinetic energy is stored by the mass and potential energy is stored by the spring. Second-order systems containing only a single type of energy storage are persistently overdamped. Although governed by second-order equations that have mathematical expressions for "natural frequency," no oscillatory behavior is possible. What is the meaning of natural frequency in this case? Karnopp and Fisher [11] note that for this case the motion can be viewed as harmonic motion in which the damping is so heavy that the motion never completes a half cycle. But, if there is no return motion, then it is not cyclic and there is no frequency.

The explanation of the natural frequency as the frequency at which an unforced second-order system oscillates in the absence of dissipation is not consistent with the systems presented here. In the examples dissipative terms appear in expressions for natural frequency. There remains the question of an unambiguous definition of natural frequency that specifies what, if any, explicit role dissipative parameters may play.

Clarity of the definition of natural frequency seems warranted. We put this question of an appropriate definition to the pedagogical community and welcome your input. We invite conference participants and readers of this manuscript to send the authors their interpretations and definitions. We propose collecting these and presenting them at ASEE 2019.

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[2]The authors acknowledge that the exercise is challenging and requires some mathematical rigor. A former MSOE undergraduate mechanical engineering student, Elise Strobach, who is currently pursuing doctoral studies at MIT, successfully solved for the limits. Most students do not take on the challenge.

Conclusions

The paper explores limiting behavior of lumped-parameter, second-order systems and shows how expressions for second-order parameters reduce to first-order time constants in the limits. Expressions for a natural frequency appear to exist for any second-order system, whether or not there are two types of energy storage elements for which oscillatory behavior can occur. Examples in the paper show that they can contain dissipative properties. A generalized, unambiguous definition of natural frequency amenable to undergraduates is sought with input from readers and the scholarly community at large.

References

[1] Close, C.M. and Frederick, D.K., *Modeling and Analysis of Dynamic Systems*, Wiley, 2001.

[2] Esfandiari, R. and Lu, B., *Modeling and Analysis of Dynamic Systems*, 2nd ed., CRC Press, 2014.

[3] Kelly, S.G., *System Dynamics and Response,* Thomson Canada, 2007.

[4] Ogata, K., *System Dynamics*, 4th ed., Pearson, 2003.

[5] Palm, W., *System Dynamics*, 3rd ed., McGraw-Hill, 2013.

[6] The Physics Forum, https://www.physicsforums.com/threads/damping-resonance-and-natural-frequency.343011/, 2009.

[7] Kerner, D.R., "Solving Windkessel Models with MLAB," http://www.civilized.com/mlabexamples/windkesmodel/htmld/, 2012.

[8] Rowell, D. "Review of First- and Second-Order System Response, MIT, 2004.

[9] Benaroya, H., Nagurka, M.L. and Han, S., *Mechanical Vibration: Analysis, Uncertainties, and Control,* 4th ed., CRC Press, 2018.

[10] Rao, S., *Mechanical Vibrations*, 6th ed., Pearson, 2016.

[11] Karnopp, B.H. and Fisher, F.E., "On the Vibrations of Overdamped Systems," *The Franklin Institute,* Vol. 327, No. 4, 1990, pp. 601-609.